

doi: 10.3969/j.issn.0490-6756.2018.05.002

单位区间典则模糊序中的平坦理想

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摘要: 本文利用连续三角模的序和分解定理给出了赋予典则模糊序的单位区间中的平坦理想的刻画. 该刻画有助于进一步研究模糊序的拓扑性质和 domain 性质.

关键词: 模糊集; 连续三角模; 模糊序; 模糊下集; 平坦理想

中图分类号: O153.1 **文献标识码:** A **文章编号:** 0490-6756(2018)05-0905-07

Flat ideals in unit interval with canonical fuzzy order

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Abstract: A characterization of flat ideals in the unit interval with the canonical fuzzy order is obtained with the help of the ordinal sum decomposition of continuous t-norms. This characterization will be useful in the study of topological and domain theoretic properties of fuzzy orders.

Keywords: Fuzzy set; Continuous triangular norm; Fuzzy order; Fuzzy lower set; Flat ideal (2010 MSC 03E72, 06A99, 03B52)

1 Introduction

A commutative quantale $[1]$ is a commutative monoid $(Q, \&, 1)$ such that Q is a complete lattice and

$$p \& (\bigvee_{j \in J} q_j) = \bigvee_{j \in J} p \& q_j$$

for all $p \in Q$ and $\{q_j\}_{j \in J} \subseteq Q$. The unit 1 of the monoid $(Q, \&, 1)$ is not necessarily the top element of Q . Given a commutative quantale $Q = (Q, \&, 1)$, since the semigroup operation $\&$ distributes over arbitrary joins, it determines a binary operation \rightarrow on Q via the adjoint property

$$p \& q \leq r \Leftrightarrow q \leq p \rightarrow r.$$

The binary operation \rightarrow is called the *implication* corresponding to $\&$. So, commutative quantales

are often the table of truth-values in many valued logic, with $\&$ playing the role of the connective “conjunction”, \rightarrow playing the role of the connective “implication”.

A Q -order (or an order valued in the quantale Q)^[2,3] on a set X is a reflexive and transitive Q -relation on X . Explicitly, a Q -order on X is a map $P: X \times X \rightarrow Q$ such that $P(x, x) \geq 1$ and $P(y, z) \& P(x, y) \leq P(x, z)$ for any $x, y, z \in X$. The pair (X, P) is called a Q -ordered set. As usual, we write X for the pair (X, P) and $X(x, y)$ for $P(x, y)$ if no confusion would arise.

If $P: X \times X \rightarrow Q$ is a Q -order on X , then $P^{\text{op}}: X \times X \rightarrow Q$, given by $P^{\text{op}}(x, y) = P(y, x)$, is also a Q -order on X (by commutativity of $\&$), called the opposite of P .

收稿日期: 2018-01-22

基金项目: 国家自然科学基金 (11771310)

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If we let $d_L(p, q) = p \rightarrow q$ for all $p, q \in Q$, then d_L is a Q -order on Q , called the canonical Q -order on Q . The opposite d_R of d_L is given by $d_R(p, q) = q \rightarrow p$. Both of (Q, d_L) and (Q, d_R) play important roles in the theory of Q -ordered sets.

A fuzzy lower set of a Q -ordered set X is a map $\varphi: X \rightarrow Q$ such that $\varphi(x) \& X(y, x) \leq \varphi(y)$ for all $x, y \in X$. Dually, a fuzzy upper set of X is a map $\psi: X \rightarrow Q$ such that $X(x, y) \& \psi(x) \leq \psi(y)$ for all $x, y \in X$.

Given a fuzzy lower set φ and a fuzzy upper set ψ of a Q -ordered set X , the tensor product of ψ and φ is defined to be the value

$$\varphi \otimes \psi = \bigvee_{x \in X} \varphi(x) \& \psi(x).$$

Intuitively, $\varphi \otimes \psi$ measures the degree that φ intersects with ψ .

Definition 1.1^[4,5] A flat ideal in a Q -ordered X is a fuzzy lower set φ of X such that

- (i) φ is inhabited in the sense that $\bigvee_{a \in X} \varphi(a) \geq 1$;
 - (ii) φ is flat in the sense that for any fuzzy upper sets ψ_1, ψ_2 of X ,
- $$\varphi \otimes (\psi_1 \wedge \psi_2) = (\varphi \otimes \psi_1) \wedge (\varphi \otimes \psi_2).$$

Flat ideals are a counterpart of directed lower sets, so, they play a crucial role in the study of the topological and domain theoretic properties of fuzzy orders. It should be noted that there exist different approaches to the notion of "fuzzy ideals", a comparative study can be found in Ref. [5]. We would like to note that the notion of flat ideals has also been extended to partial metric spaces^[6] under the name "flat left module" in Ref. [7].

In order to understand the structure of flat ideals in fuzzy ordered sets, the first step is to characterize the flat ideals in the table of truth-values, i. e. , the flat ideals in Q with the canonical fuzzy order. Among the best known commutative quantales are the unit interval $[0, 1]$ together with a continuous triangular norm (t-norm, for short). The importance of such quantales in fuzzy set theory cannot be over estimated. In particular,

the BL-logic of Hájek^[8], a very successful theory of fuzzy logic, is a logic based on such quantales.

The aim of this note is to characterize, with help of the ordinal sum decomposition of continuous t-norms, the flat ideals in the unit interval with the canonical fuzzy order. The result is very likely to be useful in the theories of topological spaces and partially ordered sets based on continuous t-norms.

2 Fuzzy lower sets of $([0, 1], d_L)$

A left continuous t-norm^[9] on the unit interval $[0, 1]$ is a binary operation $\&$ on $[0, 1]$ such that $([0, 1], \&, 1)$ is a commutative quantale. A left continuous t-norm $\&$ is said to be continuous if it is a continuous map from $[0, 1]^2$ to $[0, 1]$ with respect to the usual topology.

Example 2.1 Basic continuous t-norms and their corresponding implications.

- (i) The t-norm min:
- $$a \& b = a \wedge b = \min\{a, b\},$$

$$a \rightarrow b = \begin{cases} 1, & a \leq b, \\ b, & a > b. \end{cases}$$

The implication of the t-norm min is known as the Gödel implication.

- (ii) The product t-norm:
- $$a \& b = ab, a \rightarrow b = \begin{cases} 1, & a \leq b, \\ b/a, & a > b. \end{cases}$$

The implication of the product t-norm is known as the Goguen implication and, as a binary operation, it is continuous except at the point $(0, 0)$.

- (iii) The Łukasiewicz t-norm:
- $$a \& b = \max\{0, a + b - 1\},$$
- $$a \rightarrow b = \min\{1 - a + b, 1\}.$$

The implication of the Łukasiewicz t-norm is known as the Łukasiewicz implication and it is a continuous binary operation.

The following conclusion is essentially Proposition 2.3 in Ref. [9].

Proposition 2.2 Let $\&$ be a continuous t-norm on $[0, 1]$ and c be an idempotent element of $\&$. Then $x \& y = x \wedge y$ whenever $x \leq c \leq y$.

As an immediate corollary of the above prop-

osition we obtain that if $\&$ is a continuous t-norm and a, b ($a < b$) are idempotent elements of $\&$, then the restriction of $\&$ on $[a, b]$ is a continuous t-norm on $[a, b]$, hence $([a, b], \&, b)$ is a commutative quantale.

The following result, known as the ordinal sum decomposition of continuous t-norms, is of fundamental importance in the theory of continuous t-norms.

Theorem 2.3^[10] For each continuous t-norm $\&$ on $[0, 1]$, there exists a family of pairwise disjoint open intervals $\{(a_j, b_j)\}_{j \in J}$ of $[0, 1]$ such that

(i) for each $j \in J$, both a_j and b_j are idempotent and the restriction of $\&$ on $[a_j, b_j]$ is either isomorphic to the Łukasiewicz t-norm or to the product t-norm;

(ii) $x\&y = x \wedge y$ if $(x, y) \notin \cup_j [a_j, b_j]^2$.

Therefore, for a continuous t-norm $\&$ on $[0, 1]$ and $c \in [0, 1]$, if we let

$$c^+ = \inf\{x \in [0, 1] \mid x \geq c, x\&x = x\},$$

$$c^- = \sup\{x \in [0, 1] \mid x \leq c, x\&x = x\},$$

then

- Both c^+ and c^- are idempotent,
- If c is idempotent then $c = c^+ = c^-$,
- For each non-idempotent element c , the

restriction of $\&$ on $[c^-, c^+]$ is either isomorphic to the Łukasiewicz t-norm or to the product t-norm.

Furthermore, the implication operation in the quantale $([c^-, c^+], \&, c^+)$ is given by

$$x \rightarrow^c y = \min\{c^+, x \rightarrow y\}$$

for all $x, y \in [c^-, c^+]$.

The purpose of this section is to characterize the fuzzy lower sets and fuzzy upper sets in $([0, 1], d_L)$. In the case that the t-norm is one of the basic continuous t-norms, such characterizations have been presented in Ref. [11](Proposition 2.4 and Proposition 2.6). We use the ordinal sum decomposition of continuous t-norms to establish the desired characterizations in the general case (Proposition 2.5 and Proposition 2.7).

Proposition 2.4^[11] (fuzzy lower sets of $([0, 1], d_L)$, basic cases) Let $\varphi: [0, 1] \rightarrow [0, 1]$ be

a map.

(i) If $\&$ is the Łukasiewicz t-norm, then φ is a fuzzy lower set of $([0, 1], d_L)$ if and only if it is decreasing and 1-Lipschitz;

(ii) If $\&$ is the product t-norm, then φ is a fuzzy lower set of $([0, 1], d_L)$ if and only if it is decreasing and $y/x \leq \varphi(x)/\varphi(y)$ whenever $x > y$;

(iii) If $\&$ is the t-norm min, then φ is a fuzzy lower set of $([0, 1], d_L)$ if and only if it is decreasing and for all $x \in [0, 1], \varphi(x) \leq x$ implies that $\varphi(x) = \varphi(1)$.

Proposition 2.5 (Fuzzy lower sets of $([0, 1], d_L)$, general case) Let $\&$ be a continuous t-norm on $[0, 1]$. Then a map $\varphi: [0, 1] \rightarrow [0, 1]$ is a fuzzy lower set of $([0, 1], d_L)$ if and only if φ satisfies the following conditions:

(L1) If $a \leq b$ then $\varphi(a) \geq \varphi(b)$;

(L2) If $\varphi(c) \leq c^-$ then $\varphi(c) = \varphi(1)$;

(L3) If c is not idempotent and $\varphi(c^-) \geq c^-$, then $\varphi(x) \geq c^-$ for all $x \in [c^-, c^+]$ and the correspondence $x \rightarrow c^+ \wedge \varphi(x)$ defines a fuzzy lower set of the fuzzy ordered set $([c^-, c^+], d_L^c)$ valued in the quantale $([c^-, c^+], \&, c^+)$, where $d_L^c(x, y) = x \rightarrow^c y = \min\{c^+, x \rightarrow y\}$;

(L4) If c is idempotent and $\varphi(c) \geq c$ then $\varphi(1) \geq c$.

Proof We prove the necessity first.

(L1) Obvious.

(L2) This follows from (L1) and that $\varphi(c) = \varphi(c) \wedge c^- = \varphi(c)\&c^- \leq \varphi(c)\&(1 \rightarrow c) \leq \varphi(1)$.

(L3) For each $x \in [c^-, c^+]$, since $c^- \leq \varphi(c^-)$, then

$$c^- = c^- \&\varphi(c^-) \leq (x \rightarrow c^-) \&\varphi(c^-) \leq \varphi(x).$$

Hence $\varphi(x) \geq c^-$ for all $x \in [c^-, c^+]$. It remains to check that $d_L^c(\sigma(a), \sigma(b)) \geq d_L^c(b, a)$ whenever $c^+ \geq b > a \geq c^-$. We calculate

$$d_L^c(\sigma(a), \sigma(b)) =$$

$$\min\{c^+, d_L(\sigma(a), \sigma(b))\} =$$

$$\min\{c^+, d_L(c^+ \wedge \varphi(a), c^+ \wedge \varphi(b))\} \geq$$

$$\min\{c^+, d_L(\varphi(a), \varphi(b))\} \geq$$

$$\min\{c^+, d_L(b, a)\} =$$

$$d_L^c(b, a).$$

(L4) Since $c \leq \varphi(c)$, then $c = c\&\varphi(c) = (1$

$\rightarrow c) \& \cdot \varphi(c) \leq \varphi(1)$.

Now we prove the sufficiency. That is, $\varphi(a) \rightarrow \varphi(b) \geq b \rightarrow a$ whenever $b > a$.

Case 1. There exists some c such that $c^- \leq a < b \leq c^+$. If $\varphi(c^-) \leq c^-$, then $\varphi(c^-) = \varphi(a) = \varphi(b) = \varphi(1)$ by (L2), hence $\varphi(a) \rightarrow \varphi(b) = 1 \geq b \rightarrow a$. If $\varphi(c^-) > c^-$ then, by (L3), $\varphi(a), \varphi(b) \geq c^-$ and

$$\begin{aligned} \min\{c^+, \varphi(a) \rightarrow \varphi(b)\} &= d_L^c(\sigma(a), \sigma(b)) \geq \\ d_L^c(b, a) &= b \rightarrow a, \end{aligned}$$

hence $\varphi(a) \rightarrow \varphi(b) \geq b \rightarrow a$.

Case 2. There exists some idempotent element c such that $b > c > a$. It is clear that $b \rightarrow a = a$. Then we proceed with three subcases.

Subcase 1. $\varphi(a) \leq a^-$. Then $\varphi(a) = \varphi(b) = \varphi(1)$ by (L2), hence $\varphi(a) \rightarrow \varphi(b) = 1 \geq b \rightarrow a$.

Subcase 2. $\varphi(b) \geq a$. Then $\varphi(a) \rightarrow \varphi(b) \geq \varphi(b) \geq a$.

Subcase 3. $\varphi(a) > a^-$ and $\varphi(b) < a$. First, we show that a is not idempotent and $\varphi(a^+) < a^+$. If either a is idempotent or $\varphi(a^+) \geq a^+$, then, by (L4), we have either $\varphi(1) \geq a$ or $\varphi(1) \geq a^+$, contradicting $\varphi(b) < a$. This shows that a is not idempotent and $\varphi(a^+) < a^+$. Then, by (L2), we have $\varphi(a^+) = \varphi(1)$, hence $\varphi(a^+) = \varphi(b) < a$. Now, applying (L3) to a we obtain that

$$\begin{aligned} a = d_L^a(a^+, a) &\leq d_L^a(\varphi(a) \wedge a^+, \varphi(a^+)) = \\ \varphi(a^+) \vee \min\{a^+, \varphi(a) \rightarrow \varphi(a^+)\}, \end{aligned}$$

hence $\varphi(a) \rightarrow \varphi(b) = \varphi(a) \rightarrow \varphi(a^+) \geq a$.

The following characterization of fuzzy upper sets of $([0, 1], d_L)$ (Proposition 2.7), will be used in the next section. Since its proof is similar to that of Proposition 2.5, the details are omitted.

Proposition 2.6^[11] (fuzzy upper sets of $([0, 1], d_L)$, basic cases) Let $\psi: [0, 1] \rightarrow [0, 1]$ be a map.

(i) If $\& \cdot$ is the Łukasiewicz t-norm, then ψ is a fuzzy upper set of $([0, 1], d_L)$ if and only if it is increasing and 1-Lipschitz;

(ii) If $\& \cdot$ is the product t-norm, then ψ is a fuzzy upper set of $([0, 1], d_L)$ if and only if it is increasing and $y/x \leq \psi(y)/\psi(x)$ whenever $x > y$;

(iii) If $\& \cdot$ is the t-norm min, then ψ is a fuzzy upper set of $([0, 1], d_L)$ if and only if it is increasing and for every $x \in [0, 1], \psi(x) < x$ implies $\psi(x) = \psi(1)$.

Proposition 2.7 (Fuzzy upper sets in $([0, 1], d_L)$, general case) A map $\psi: [0, 1] \rightarrow [0, 1]$ is a fuzzy upper set of $([0, 1], d_L)$ if and only if ψ satisfies the following conditions:

(U1) If $a \leq b$ then $\psi(a) \leq \psi(b)$;

(U2) If $\psi(c) < c^-$ then $\psi(c) = \psi(1)$;

(U3) If c is not idempotent and $\psi(c^-) \geq c^-$, then $\psi(x) \geq c^-$ for all $x \in [c^-, c^+]$ and the correspondence $x \rightarrow c^+ \wedge \psi(x)$ defines a fuzzy upper set of the fuzzy ordered set $([c^-, c^+], d_L^c)$ valued in the quantale $([c^-, c^+], \& \cdot, c^+)$, where $d_L^c(x, y) = x \rightarrow^c y = \min\{c^+, x \rightarrow y\}$.

3 Flat ideals in $([0, 1], d_L)$

In this section, we characterize flat ideals in $([0, 1], d_L)$. The strategy is to do this in the special case that $\& \cdot$ is one of the basic continuous t-norms, then in the general case with help of the ordinal sum decomposition of continuous t-norms.

Given a fuzzy ordered set X , a net $\{x_\lambda\}_{\lambda \in D}$ in X is forward Cauchy^[3] if

$$\bigvee_{\lambda \in D} \bigwedge_{\sigma \geq \mu \geq \lambda} X(x_\mu, x_\sigma) = 1.$$

A fuzzy lower set φ of a fuzzy ordered set X is a forward Cauchy ideal if there exists some forward Cauchy net $\{x_\lambda\}_{\lambda \in D}$ such that

$$\varphi = \bigvee_{\lambda \in D} \bigwedge_{\mu \geq \lambda} X(-, x_\mu).$$

The following theorem was proved in Ref. [4] in the case that $\& \cdot$ is isomorphic to the product t-norm, and in Ref. [12] in the case that $\& \cdot$ is isomorphic to the Łukasiewicz t-norm.

Theorem 3.1 Let $\& \cdot$ be a continuous t-norm on $[0, 1]$ that is either isomorphic to the product t-norm or to the Łukasiewicz t-norm, and Q be the quantale $([0, 1], \& \cdot, 1)$. Then for each Q -ordered set X , flat ideals coincide with forward Cauchy ideals.

Theorem 3.2 (Flat ideals in $([0, 1], d_L)$, basic cases) (i) If $\&$ is a continuous t-norm on $[0, 1]$ that is either isomorphic to the Łukasiewicz t-norm or to the product t-norm, then for every flat ideal φ of $([0, 1], d_L)$, there exists some $x \in [0, 1]$ such that $\varphi = d_L(-, x)$;

(ii) If $\&$ is the t-norm min, then a fuzzy lower set φ of $([0, 1], d_L)$ is a flat ideal if and only if $\varphi(0) = 1$.

Proof (i) By Theorem 3.1,

$$\varphi(x) = \bigvee_{\lambda \in D} \bigwedge_{\mu \geq \lambda} (x \rightarrow x_\mu)$$

for some Cauchy net $\{x_\lambda\}_{\lambda \in I}$. Then $x = \bigvee_{\lambda \in D} \bigwedge_{\mu \geq \lambda} x_\mu$ satisfies the requirement.

(ii) Trivial.

Theorem 3.3 (Flat ideals in $([0, 1], d_L)$, general case) Let $\&$ be a continuous t-norm on $[0, 1]$. Then a fuzzy lower set φ of $([0, 1], d_L)$ is a flat ideal if and only if it satisfies:

(F1) $\varphi(0) = 1$;

(F2) For each $c \in [0, 1]$, if $\varphi(c) \geq c^+$ then $\varphi(c)$ is idempotent;

(F3) For each $c \in [0, 1]$, if c is not idempotent and $\varphi(c^-) > c^-$, then the correspondence $x \rightarrow c^+ \wedge \varphi(x)$ defines a flat ideal in the fuzzy ordered set $([c^-, c^+], d_L^c)$ valued in the quantale $([c^-, c^+], \&, c^+)$, where $d_L^c(x, y) = x \rightarrow^c y = \min\{c^+, x \rightarrow y\}$.

Proof The proof is divided into four lemmas given below. The necessity part is contained in Lemma 3.4 and Lemma 3.5; the sufficiency part follows from Lemma 3.6 and Lemma 3.7.

Lemma 3.4 Let φ be a flat ideal in $([0, 1], d_L)$. Then for each $c \in [0, 1]$, if $\varphi(c) \geq c^+$ then $\varphi(c)$ is idempotent.

Proof Consider the fuzzy upper sets ψ_1, ψ_2 of $([0, 1], d_L)$ given by $\psi_1 \equiv \varphi(c)$ and $\psi_2 = c \rightarrow id$. Since $\varphi(0) = 1$, then $\varphi \otimes \psi_1 = \varphi(c)$. Since φ is fuzzy lower set of $([0, 1], d_L)$, it follows that

$$\begin{aligned} \varphi \otimes \bar{\psi} &= \bigvee_{x \in [0, 1]} \varphi(x) \& \bar{\psi}(x) = \\ & \left(\bigvee_{x \in [0, c^-)} \varphi(x) \& x \right) \vee \left(\bigvee_{x \in [c^-, c^+]} \varphi(x) \& \psi(x) \right) \vee \left(\bigvee_{x \in (c^+, 1]} \varphi(x) \& \psi(c^+) \right) \leq \end{aligned}$$

$$\varphi \otimes \psi_2 = \bigvee_{x \in [0, 1]} \varphi(x) \& (c \rightarrow x) = \varphi(c).$$

Hence $(\varphi \otimes \psi_1) \wedge (\varphi \otimes \psi_2) = \varphi(c)$.

Since $\varphi(c) \geq c^+$, then

$$(\psi_1 \wedge \psi_2)(x) = \begin{cases} c \rightarrow x, & 0 \leq x < c, \\ \varphi(c), & c \leq x \leq 1, \end{cases}$$

hence

$$\begin{aligned} \varphi(c) &= (\varphi \otimes \psi_1) \wedge (\varphi \otimes \psi_2) = \\ & (\varphi \otimes \psi_1 \wedge \psi_2) = \end{aligned}$$

$$\left(\bigvee_{x \in [0, c)} \varphi(x) \& (c \rightarrow x) \right) \vee$$

$$\left(\bigvee_{x \in [c, 1]} \varphi(x) \& \varphi(c) \right) \leq$$

$$c^+ \vee (\varphi(c) \& \varphi(c)) = \varphi(c) \& \varphi(c)$$

showing that $\varphi(c)$ is idempotent.

Lemma 3.5 Let φ be a flat ideal in $([0, 1], d_L)$. For each $c \in [0, 1]$, if c is not idempotent and $\varphi(c^-) > c^-$ then $\sigma(x) = c^+ \wedge \varphi(x)$ is a flat ideal in the fuzzy ordered set $([c^-, c^+], d_L^c)$ valued in $([c^-, c^+], \&, c^+)$, where $d_L^c(x, y) = x \rightarrow^c y = \min\{c^+, x \rightarrow y\}$.

Proof Since $\varphi(c^-) > c^-$, then $\varphi(c^-)$ is idempotent by Lemma 3.4. Thus $\varphi(c^-) \geq c^+$ and then $\sigma(c^-) = c^+$, showing that σ is inhabited. It remains to show that for any fuzzy upper sets ψ_1, ψ_2 of the fuzzy ordered set $([c^-, c^+], d_L^c)$ (valued in $([c^-, c^+], \&, c^+)$),

$$\sigma \otimes (\psi_1 \wedge \psi_2) = (\sigma \otimes \psi_1) \wedge (\sigma \otimes \psi_2).$$

Step 1. For each fuzzy upper set ψ in $([c^-, c^+], d_L^c)$, the map $\bar{\psi}: [0, 1] \rightarrow [0, 1]$, given by

$$\bar{\psi}(x) = \begin{cases} x, & x \in [0, c^-), \\ \psi(x), & x \in [c^-, c^+], \\ \psi(c^+), & x \in (c^+, 1], \end{cases}$$

is a fuzzy upper set of $([0, 1], d_L)$. This follows from Proposition 2.7 immediately.

Step 2. For each fuzzy upper set ψ in $([c^-, c^+], d_L^c)$, $\sigma \otimes \psi = \varphi \otimes \bar{\psi}$. We calculate

$$c^- \vee \left(\bigvee_{x \in [c^-, c^+]} (\varphi(x) \wedge c^+) \&\cdot \psi(x) \right) \vee (\varphi(c^+) \&\cdot \psi(c^+)) = \bigvee_{x \in [c^-, c^+]} \sigma(x) \&\cdot \psi(x) = \sigma \otimes \psi.$$

Step 3. For any fuzzy upper sets ψ_1, ψ_2 of $([c^-, c^+], d_L^c)$,

$$\begin{aligned} \sigma \otimes (\psi_1 \wedge \psi_2) &= \varphi \otimes \overline{(\psi_1 \wedge \psi_2)} = \\ &= \varphi \otimes (\overline{\psi_1} \wedge \overline{\psi_2}) = \\ &= (\varphi \otimes \overline{\psi_1}) \wedge (\varphi \otimes \overline{\psi_2}) = \\ &= (\sigma \otimes \psi_1) \wedge (\sigma \otimes \psi_2). \end{aligned}$$

Therefore, σ is a flat ideal in the fuzzy ordered set $([c^-, c^+], d_L^c)$ valued in the quantale $([c^-, c^+], \&\cdot, c^+)$.

The verification of the following lemma is straightforward and is left to the reader.

Lemma 3.6 Let K be a subset of $[0, 1]$, $\varphi: K \rightarrow [0, 1]$ be a decreasing map, and $\psi_1, \psi_2: K \rightarrow [0, 1]$ be increasing maps. Then

$$\begin{aligned} \bigvee_{x \in K} \varphi(x) \wedge \psi_1(x) \wedge \psi_2(x) &= \\ \left(\bigvee_{x \in K} \varphi(x) \wedge \psi_1(x) \right) \wedge \left(\bigvee_{x \in K} \varphi(x) \wedge \psi_2(x) \right). \end{aligned}$$

Lemma 3.7 Suppose φ is a fuzzy lower set of $([0, 1], d_L)$ that satisfies (F2) and (F3) in Theorem 3.3. Let $K_\varphi = \{x \in [0, 1] \mid \varphi(x) \geq x^+\}$. Then for each fuzzy upper set ψ of $([0, 1], d_L)$,

$$\varphi \otimes \psi = \bigvee_{x \in K_\varphi} \varphi(x) \wedge \psi(x).$$

Proof For all $x \in K_\varphi$, $\varphi(x)$ is idempotent by (F2). Thus, $\varphi(x) \&\cdot \psi(x) = \psi(x) \wedge \varphi(x)$ whenever $x \in K_\varphi$. Since

$$\varphi \otimes \psi = \bigvee_{x \in [0, 1]} \varphi(x) \&\cdot \psi(x),$$

it suffices to show that for all $c \notin K_\varphi$, there is some $b \in K_\varphi$ such that $\varphi(c) \&\cdot \psi(c) \leq \varphi(b) \wedge \psi(b)$. We proceed with three cases.

Case 1. c is non-idempotent and $\varphi(c^-) > c^-$. By (F3), $\sigma(x) = c^+ \wedge \varphi(x)$ is a flat ideal in the fuzzy ordered set $([c^-, c^+], d_L^c)$. Since the restriction of $\&\cdot$ on $[c^-, c^+]$ is either isomorphic to the Łukasiewicz t-norm or to the product t-norm, by Theorem 3.2 there is some $b \in [c^-, c^+]$ such that $\sigma(x) = d_L^c(x, b)$. We claim that this b satisfies the

requirement, that is, $\varphi(c) \&\cdot \psi(c) \leq \varphi(b) \wedge \psi(b)$.

If $c \leq b$, then $\sigma(c) = d_L^c(c, b) = c^+$, hence $\varphi(c) \geq c^+$, showing that $c \in K_\varphi$, contradictory to that $c \notin K_\varphi$. Hence, we have $b < c < c^+$ and $\varphi(c) < c^+$. Consequently, $\varphi(c) = \sigma(c) = c \rightarrow b$. Then

$$\varphi(c) \&\cdot \psi(c) \leq (c \rightarrow b) \leq c^+ = \sigma(b) \leq \varphi(b)$$

and, since ψ is a fuzzy upper set,

$$\varphi(c) \&\cdot \psi(c) = (c \rightarrow b) \&\cdot \psi(c) \leq \psi(b).$$

Hence $\varphi(c) \&\cdot \psi(c) \leq \varphi(b) \wedge \psi(b)$, as desired.

Case 2. c is non-idempotent and $\varphi(c^-) \leq c^-$. We distinguish two subcases.

Subcase 1. $\varphi(c)$ is idempotent. We show that $b = \varphi(c)$ satisfies the requirement. Since $\varphi(c^-) \leq c^-$, then $b = \varphi(c) \leq \varphi(c^-) \leq c^- < c$, hence $\varphi(b) \geq \varphi(c) = b = b^+$, showing that $b \in K_\varphi$. If $\psi(b) \geq b$, then $\varphi(b) \&\cdot \psi(b) \geq b \&\cdot b = b \geq \varphi(c) \&\cdot \psi(c)$. If $\psi(b) < b$, then $\psi(b) = \psi(c)$ by (U2) in Proposition 2.7, hence $\varphi(b) \&\cdot \psi(b) \geq \varphi(c) \&\cdot \psi(c)$. So, b satisfies the requirement.

Subcase 2. $\varphi(c)$ is non-idempotent. Since $\varphi(c^-) \leq c^-$, by (L2) in Proposition 2.5 we have $\varphi(c^-) = \varphi(1)$, hence $\varphi(c) = \varphi(1)$. Now we use this fact to show that $\varphi(\varphi(c^-)) > \varphi(c^-)$. Since $\varphi(\varphi(c^+)) \rightarrow \varphi(1) \geq 1 \rightarrow \varphi(c^+) = \varphi(c^+)$, it follows that

$$\begin{aligned} \varphi(c^+) \&\cdot \varphi(\varphi(c^+)) &= \\ \varphi(c^+) \wedge \varphi(\varphi(c^+)) &\leq \varphi(1), \end{aligned}$$

hence $\varphi(\varphi(c^+)) \leq \varphi(1)$ because $\varphi(c^+) > \varphi(c) = \varphi(1)$, so $\varphi(\varphi(c^+)) = \varphi(1) = \varphi(c)$. Thus

$$\varphi(\varphi(c^-)) \geq \varphi(\varphi(c^+)) = \varphi(c) > \varphi(c^-).$$

By (F3), $\sigma(x) = \varphi(c^+) \wedge \varphi(x)$ is a flat ideal in the fuzzy ordered set $([\varphi(c)^-, \varphi(c)^+], d_L^{\varphi(c)})$ (valued in $([\varphi(c)^-, \varphi(c)^+], \&\cdot, \varphi(c^+))$). Then there is some $b \in [\varphi(c)^-, \varphi(c)^+]$ such that $\sigma(x) = d_L^{\varphi(c)}(x, b)$. Since $\varphi(\varphi(c^+)) = \varphi(c)$, it follows that $\sigma(\varphi(c^+)) = \varphi(c)$, hence $b < \varphi(c)^+$. We claim that this b satisfies the requirement. It suffices to show that $b \in K_\varphi$ and

$$\begin{aligned} \varphi(c) \&\cdot \psi(c) &\leq \varphi(\varphi(c^+)) \&\cdot \psi(\varphi(c^+)) \leq \\ &\varphi(b) \wedge \psi(b). \end{aligned}$$

Firstly, since $\sigma(b) = \varphi(c)^+$, then $\varphi(b) \geq \varphi(c)^+ = b^+$, hence $b \in K_\varphi$.

Secondly, we check the inequality

$$\varphi(c) \&\psi(c) \leq \varphi(\varphi(c)^+) \&\psi(\varphi(c)^+).$$

If $\psi(\varphi(c)^+) \geq \varphi(c)^+$, then

$$\varphi(\varphi(c)^+) \&\psi(\varphi(c)^+) = \varphi(c) \geq \varphi(c) \&\psi(c).$$

If $\psi(\varphi(c)^+) < \varphi(c)^+$, then by (U2) in Proposition 2.7 we have $\psi(\varphi(c)^+) = \psi(c)$, hence

$$\varphi(\varphi(c)^+) \&\psi(\varphi(c)^+) \geq \varphi(c) \&\psi(c).$$

Thirdly, we check the inequality

$$\varphi(\varphi(c)^+) \&\psi(\varphi(c)^+) \leq \varphi(b) \wedge \psi(b).$$

Since $\varphi(\varphi(c)^+) = \varphi(c)$, it follows that $\varphi(\varphi(c)^+) = \sigma(\varphi(c)^+) = \varphi(c)^+ \rightarrow b$, then

$$\begin{aligned} \varphi(\varphi(c)^+) \&\psi(\varphi(c)^+) &= \\ (\varphi(c)^+ \rightarrow b) \&\psi(\varphi(c)^+) &\leq \psi(b). \end{aligned}$$

Since

$$\begin{aligned} \varphi(\varphi(c)^+) \&\psi(\varphi(c)^+) &= \varphi(c) \&\psi(\varphi(c)^+) \leq \\ \varphi(c) &\leq \varphi(c)^+ = \sigma(b) \leq \varphi(b), \end{aligned}$$

then the desired inequality follows.

Case 3. c is idempotent. We also distinguish two subcases.

Subcase 1. $\varphi(c)$ is idempotent. We claim that $b = \varphi(c)$ satisfies the requirement. Since $c \notin K_\varphi$, then $b = \varphi(c) < c$, hence $\varphi(b) \geq \varphi(c) = b$. This shows $b \in K_\varphi$. If $\psi(b) \geq b$, then $\varphi(b) \&\psi(b) \geq b \&b = b \geq \varphi(c) \&\psi(c)$. If $\psi(b) < b$, then by (U2) in Proposition 2.7 we have that $\psi(b) = \psi(c)$, hence $\varphi(b) \&\psi(b) \geq \varphi(c) \&\psi(c)$.

Subcase 2. $\varphi(c)$ is non-idempotent. Since $c \notin K_\varphi$, then $\varphi(c) < c$, hence $\varphi(c) = \varphi(1)$ by (L2) in Proposition 2.5. The rest of the proof then pro-

ceeds in the same way as that for **Subcase 2** in Case 2, the details are omitted here.

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引用本文格式:

中文: 赖洪亮, 张德学, 张高. 单位区间典则模糊序中的平坦理想 [J]. 四川大学学报: 自然科学版, 2018, 55: 905.

英文: Lai H L, Zhang D X, Zhang G. Flat ideals in unit interval with canonical fuzzy order [J]. J Sichuan Univ: Nat Sci Ed, 2018, 55: 905.