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$3n$ 体三重嵌套正多边形中心构型问题

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摘 要：本文运用 Dziobek 方程组研究了 9 体和 12 体三重嵌套正多边形中心构型问题.对于 9 体问题，本文发现了三类新的三重嵌套正三角形中心构型. 对于 12 体问题，本文也发现了几类不同的三重嵌套正多边形中心构型.

关键词：多体问题；平面中心构型；嵌套正多边形；牛顿势
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Central configurations for planar $3n$-body problem: triple nested regular polygons

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Abstract: We study the existence of some families of triple nested planar central configurations for the n-body problem with $n=9,12$. For 9-body problem we show that there exist three families of triple nested triangular central configurations. For 12-body problem we show that there exist different families of triple nested regular polygonal central configurations in different cases.

Keywords: n-body problem; Planar central configuration; Nested regular polygon; Newtonian law

1 Introduction

The Newtonian n-body problem[1,2] concerns the motion of n mass points moving in space according to Newtonian law:

$$m_i \ddot{x}_i = -\sum_{j=1,j\neq i}^{n} \frac{m_i m_j (x_i - x_j)}{r_{ij}^3}, i=1,2,\cdots,n \tag{1}$$

Here $x_i \in \mathbf{R}^d$ is the position of mass $m_i > 0$, the gravitational constant is taken equal to 1, and $r_{ij} = |x_i - x_j|$ is the Euclidean distance between x_i and x_j. The space of configuration is defined by

$$X = \{(x_1,\cdots,x_n) \in (\mathbf{R}^d)^n : x_i \neq x_j \text{ for all } i\neq j\},$$

while the center of mass is given by

$$c = (m_1 x_1 + \cdots + m_n x_n)/M,$$

where $M = m_1 + \cdots + m_n$ is the total mass.

A configurations $x = (x_1,\cdots,x_n) \in X$ is called a central configuration[2,3] if there exists a constant $\lambda \neq 0$, called the multiplier, such that

$$-\lambda(x_i - c) = \sum_{j=1,j\neq i}^{n} \frac{m_j (x_j - x_i)}{r_{ij}^3}, i=1,\cdots,n \tag{2}$$

It is easy to see that a central configuration remains a central configuration after a rotation in $\mathbf{R}^d$ and a scalar multiplication. More precisely, let $A \in SO(d)$ and $a > 0$. If $x = (x_1,\cdots,x_n)$ is a central configuration, so are $Ax = (Ax_1,\cdots,Ax_n)$ and $ax = (ax_1,\cdots,ax_n)$. Two central configurations are said to be equivalent if one can be transformed to the other by a rotation and a scalar

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multiplication. When we say a central configuration, we mean a class of central configurations as defined by the above equivalent relation.

There are several reasons why central configurations are of special importance in the study of the n -body problem[3-7].

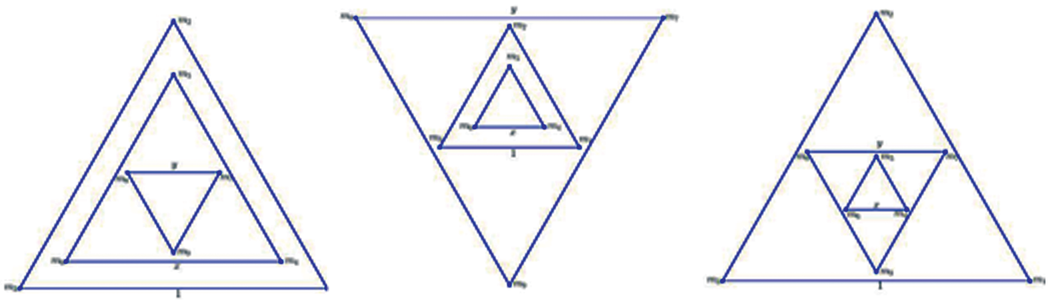

Fig. 1 Three families of triple nested triangular central configurations for 9-body problem

In this paper we are interested in planar central configurations, that is $d = 2$. The existence of double nested planar central configurations for $2n$ -bodies is known for two nested regular n -gons with common center[8,9]. Refs. [9～11] studied the necessary conditions about nested regular polyhedra. The work was completed by Corbera and Llibre[12,13]. Llibre and Mello[8] shown the existence of families of triple and quadruple nested planar central configurations for the n-body problem with $n = 6,8,9$. In this paper we find new classes of planar triple nested central configurations of n -bodies for $n = 9$ and $n = 12$ according to Fig. 1 ～Fig. 4.

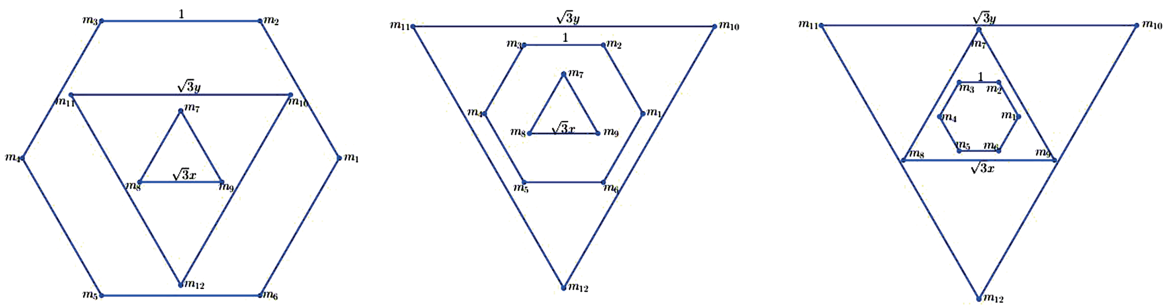

Fig. 2 Three families of triple nested regular polygonal central configuration for 12-body problem

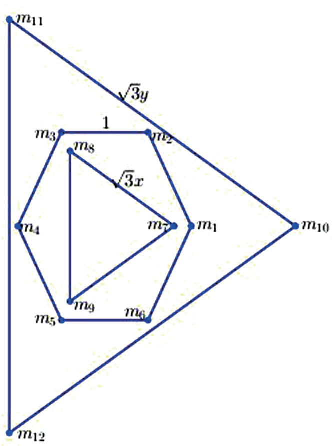

Fig. 3 One families of triple nested regular polygonal central configurations for 12-body problem

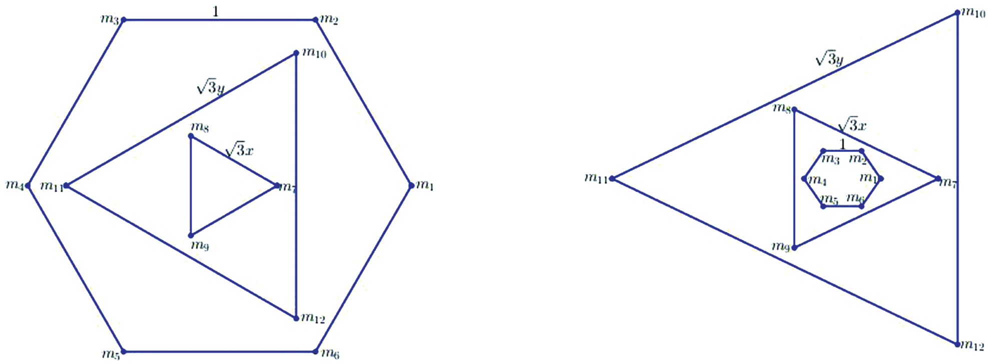

Fig. 4　Two families of triple nested regular polygonal central configurations for 12-body problem

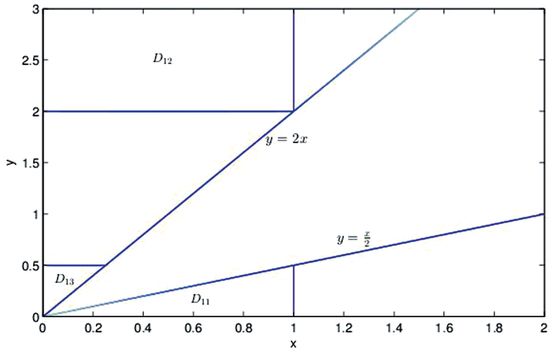

Fig. 5　The region D_1

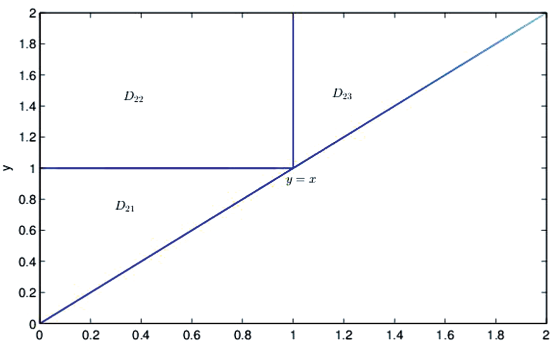

Fig. 6　The region D_2

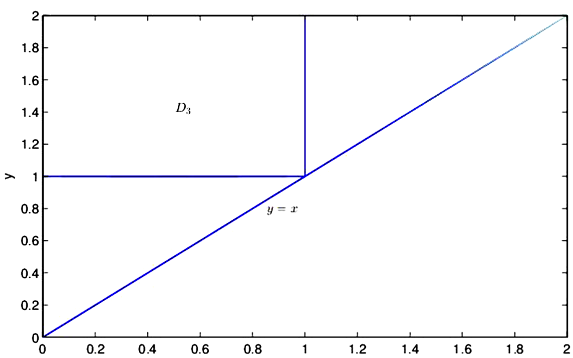

Fig. 7　The region D_3

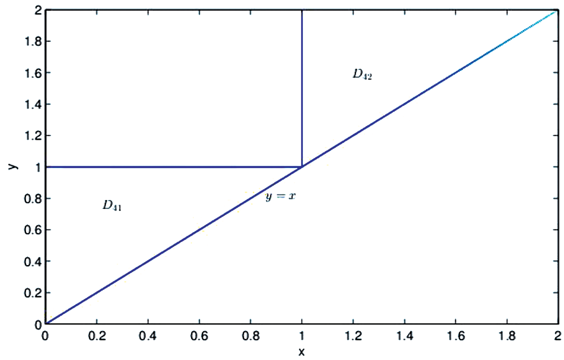

Fig. 8　The region D_4

The main results of this paper are the following.

Theorem 1.1　Assume that we have a triple of three masses $M_1=m_1=m_2=m_3$, $M_2=m_4=m_5=m_6$, and $M_3=m_7=m_8=m_9$ on the vertices of equilateral triangles with common barycenter whose side have length 1, x and y (see Fig. 1), such that $(x,y)\in D_1=D_{11}\cup D_{12}\cup D_{13}$ (see Fig. 5), where

$$D_{11}=\left\{(x,y):0<x<1,0<y<\frac{x}{2}\right\},$$

$$D_{12}=\{(x,y):0<x<1,y>2\},$$

$$D_{13}=\left\{(x,y):0<y<\frac{1}{2},0<x<\frac{y}{2}\right\}.$$

Then there exist three non-empty open sets in $D_{1i}(i=1,2,3)$ ($(x,y)\in D_{13}$ has been found in Ref. [8]), respectively, and positive masses M_1, M_2 and M_3 such that the nine bodies with these masses form three central configurations of the 9-body problem.

Theorem 1.2　Assume that we have a triple of three masses

$$M_1=m_1=m_2=m_3=m_4=m_5=m_6,$$

$M_2 = m_7 = m_8 = m_9$,

$M_3 = m_{10} = m_{11} = m_{12}$

on the vertices of a regular hexagon and two equilateral triangles with common barycenter whose side have length 1, $\sqrt{3}x$ and $\sqrt{3}y$ (see Fig. 2), such that $(x,y) \in D_2 = D_{21} \cup D_{22} \cup D_{23}$ (see Fig. 6), where

$D_{21} = \{(x,y): 0 < x < y < 1\}$,

$D_{22} = \{(x,y): 0 < x < 1, y > 1\}$,

$D_{23} = \{(x,y): 1 < x < y\}$.

Then there exist three non-empty open sets in D_{2i} $(i=1,2,3)$, respectively, and positive masses M_1, M_2 and M_3 such that the twelve bodies with these masses form three central configurations of the 12-body problem.

Theorem 1.3 Assume that we have a triple of three masses

$M_1 = m_1 = m_2 = m_3 = m_4 = m_5 = m_6$,

$M_2 = m_7 = m_8 = m_9$,

and

$M_3 = m_{10} = m_{11} = m_{12}$

on the vertices of a regular hexagon and two equilateral triangles with common barycenter whose side have length 1, $\sqrt{3}x$ and $\sqrt{3}y$ (see Fig. 3), such that $(x,y) \in D_3$ (see Fig. 7), where

$D_3 = \{(x,y): 0 < x < 1, y > 1\}$.

Then there exists a non-empty open set in D_3 and positive masses M_1, M_2 and M_3 such that the twelve bodies with these masses form one central configuration of the 12-body problem.

Theorem 1.4 Assume that we have a triple of three masses

$M_1 = m_1 = m_2 = m_3 = m_4 = m_5 = m_6$,

$M_2 = m_7 = m_8 = m_9$,

and

$M_3 = m_{10} = m_{11} = m_{12}$

on the vertices of a regular hexagon and two equilateral triangles with common barycenter whose side have length 1, $\sqrt{3}x$ and $\sqrt{3}y$ (see Fig. 4), such that $(x,y) \in D_4 = D_{41} \cup D_{42}$ (see Fig. 8), where

$D_{41} = \{(x,y): 0 < x < y < 1\}$,

$D_{42} = \{(x,y): 1 < x < y\}$.

Then there exist two non-empty open sets in D_{4i} $(i=1,2)$, respectively, and positive masses M_1, M_2 and M_3 such that the twelve bodies with these masses form two central configurations of the 12-body problem.

Without loss of generality, we suppose that $0 < x < y$ and $x \neq 1, y \neq 1$ in the next sections.

2 Proof of Theorem 1.1

For the planar central configurations, instead of working with equations (2), we consider the Dziobek equations[14]:

$$f_{ij} = \sum_{k=1, k\neq i,j}^{n} m_k (R_{ik} - R_{jk}) \Delta_{ijk} = 0 \tag{3}$$

for $1 \leqslant i < j \leqslant n$, here, $R_{ij} = 1/r_{ij}{}^3$ and $\Delta_{ijk} = (x_i - x_j) \wedge (x_i - x_k)$. Thus Δ_{ijk} gives two times the signed area of the triangle with vertices at m_i, m_j and m_k, equations (3) is a system of $n(n-2)/2$ equations.

For the 9-body problem, equations (3) is a system of 36 equations. Without loss of generality, we can assume that

$$m_1\left(\frac{1}{2}, -\frac{\sqrt{3}}{6}\right), m_2\left(0, \frac{\sqrt{3}}{3}\right), m_3\left(-\frac{1}{2}, -\frac{\sqrt{3}}{6}\right),$$

$$m_4\left(\frac{1}{2}x, -\frac{\sqrt{3}}{6}x\right), m_5\left(0, \frac{\sqrt{3}}{3}x\right),$$

$$m_6\left(-\frac{1}{2}x, -\frac{\sqrt{3}}{6}x\right), m_7\left(\begin{array}{c}\frac{1}{2}y, \\ \frac{\sqrt{3}}{6}y\end{array}\right),$$

$$m_8\left(-\frac{1}{2}y, \frac{\sqrt{3}}{6}y\right), m_9\left(0, -\frac{\sqrt{3}}{3}y\right) \tag{4}$$

where $(x,y) \in D_1$. According to Fig. 1, our class of configurations with nine bodies must satisfy

$r_{12} = r_{13} = r_{23} = 1, r_{45} = r_{46} = r_{56} = x$,

$r_{78} = r_{79} = r_{89} = y$,

$r_{14} = r_{25} = r_{36} = \frac{\sqrt{3}}{3}(1-x)$,

$r_{15} = r_{16} = r_{24} = r_{26} = r_{34} = r_{35} = \frac{\sqrt{3}}{3}\sqrt{x^2+x+1}$,

$r_{17} = r_{19} = r_{27} = r_{28} = r_{38} = r_{39} = \frac{\sqrt{3}}{3}\sqrt{y^2-y+1}$,

$r_{18} = r_{29} = r_{37} = \frac{\sqrt{3}}{3}(y+1)$,

$r_{47} = r_{49} = r_{57} = r_{58} = r_{68} = r_{69} =$

$\frac{\sqrt{3}}{3}\sqrt{x^2-xy+y^2}$,

$$r_{48}=r_{59}=r_{67}=\frac{\sqrt{3}}{3}(x+y) \tag{5}$$

Taking into account that $M_1=m_1=m_2=m_3$, $M_2=m_4=m_5=m_6$, $M_3=m_7=m_8=m_9$, the equations $f_{12}=0$, $f_{13}=0$, $f_{23}=0$, $f_{45}=0$, $f_{46}=0$, $f_{56}=0$, $f_{78}=0$, $f_{79}=0$, $f_{89}=0$, $f_{14}=0$, $f_{25}=0$, $f_{36}=0$, $f_{18}=0$, $f_{29}=0$, $f_{37}=0$, $f_{48}=0$, $f_{59}=0$ and $f_{67}=0$ of equations (3) are trivially satisfied. The other 18 equations of equations (3) can be put into 3 sets of equivalent equations:

Set 1. Equations $f_{15}=0$, $f_{16}=0$, $f_{24}=0$, $f_{26}=0$, $f_{34}=0$ and $f_{35}=0$ can be written as

$$a_{11}M_1+a_{12}M_2+a_{13}M_3=0 \tag{6}$$

where

$$a_{11}=(R_{12}-R_{14})\Delta_{152}+(R_{12}-R_{15})\Delta_{153},$$
$$a_{12}=(R_{14}-R_{45})\Delta_{154}+(R_{15}-R_{45})\Delta_{156},$$
$$a_{13}=(R_{17}-R_{47})\Delta_{157}+(R_{18}-R_{47})\Delta_{158}+(R_{17}-R_{48})\Delta_{159}.$$

Set 2. Equations $f_{17}=0$, $f_{19}=0$, $f_{27}=0$, $f_{28}=0$, $f_{38}=0$ and $f_{39}=0$ can be written as

$$a_{21}M_1+a_{22}M_2+a_{23}M_3=0 \tag{7}$$

where

$$a_{21}=(R_{12}-R_{17})\Delta_{172}+(R_{12}-R_{18})\Delta_{173},$$
$$a_{22}=(R_{14}-R_{47})\Delta_{174}+(R_{15}-R_{47})\Delta_{175}+(R_{15}-R_{48})\Delta_{176},$$
$$a_{23}=(R_{18}-R_{78})\Delta_{178}+(R_{17}-R_{78})\Delta_{179}.$$

Set 3. Equations $f_{47}=0$, $f_{49}=0$, $f_{57}=0$, $f_{58}=0$, $f_{68}=0$ and $f_{69}=0$ can be written as

$$a_{31}M_1+a_{32}M_2+a_{33}M_3=0 \tag{8}$$

where

$$a_{31}=(R_{14}-R_{17})\Delta_{471}+(R_{15}-R_{17})\Delta_{472}+(R_{15}-R_{18})\Delta_{473},$$
$$a_{32}=(R_{45}-R_{47})\Delta_{475}+(R_{45}-R_{48})\Delta_{476},$$
$$a_{33}=(R_{48}-R_{78})\Delta_{478}+(R_{47}-R_{78})\Delta_{479}.$$

By fundamental calculation, we find that equation (8) is a linear combination of equations (6) and (7). Thus equation (3) has nontrivial solutions. In order to have planar central configurations according to Fig. 1, we need to find positive solutions M_1, M_2 and M_3 of equations (6) and (7). According to assumptions (4), we have

$$a_{11}=\frac{\sqrt{3}}{2}\left[x+\frac{\sqrt{3}}{(1-x)^2}-\frac{\sqrt{3}(2x+1)}{(x^2+x+1)^{\frac{3}{2}}}\right],$$

$$a_{12}=\frac{\sqrt{3}}{2}x\left[\frac{\sqrt{3}}{(1-x)^2}+\frac{\sqrt{3}(x+2)}{(x^2+x+1)^{\frac{3}{2}}}-\frac{1}{x^3}\right],$$

$$a_{13}=\frac{3}{2}\left[\frac{x(2-y)}{(y^2-y+1)^{\frac{3}{2}}}+\frac{x}{(1+y)^2}-\frac{2x-y}{(x^2-xy+y^2)^{\frac{3}{2}}}-\frac{1}{(x+y)^2}\right],$$

$$a_{21}=\frac{\sqrt{3}}{2}\left[y-\frac{\sqrt{3}}{(1+y)^2}-\frac{\sqrt{3}(2y-1)}{(y^2-y+1)^{\frac{3}{2}}}\right],$$

$$a_{22}=\frac{3}{2}\left[\frac{y}{(1-x)^2}+\frac{y(x+2)}{(x^2+x+1)^{\frac{3}{2}}}-\frac{2y-x}{(x^2-xy+y^2)^{\frac{3}{2}}}-\frac{1}{(x+y)^2}\right],$$

$$a_{23}=\frac{\sqrt{3}}{2}y\left[\frac{\sqrt{3}}{(1+y)^2}+\frac{\sqrt{3}(2-y)}{(y^2-y+1)^{\frac{3}{2}}}-\frac{1}{y^3}\right].$$

Let $N=(n_1,n_2,n_3)=N_1\wedge N_2$, where $N_1=(a_{11},a_{12},a_{13})$ and $N_2=(a_{21},a_{22},a_{23})$ be the vector parallel to the straight line defined by the intersection of the two planes orthogonal to N_1 and N_2. Therefore there will be positive masses M_1, M_2 and M_3 solutions of equations (6) and (7) if and only if the components of the vector N have the same sign.

The case of $(x,y)\in D_{13}\subset D_1$ has been found by Llibre and Mello[8], in the rest of the proof we only show the existence of central configurations with $(x,y)\in D_{11}\cup D_{12}\subset D_1$.

Consider the point $(x,y)=(0.4,0.1)\in D_{11}\subset D_1$ and substituting $x=0.4$, $y=0.1$ into n_1, n_2 and n_3, it follows that $n_1(0.4,0.1)>0$, $n_2(0.4,0.1)>0$, $n_3(0.4,0.1)>0$. Therefore the components of the mass vector N have the same sign.

In order to give some information about the values of the masses, we consider $M_1=m_1=m_2=m_3=1$. Then

$$M_2=m_4=m_5=m_6=0.921718508801,$$
$$M_3=m_7=m_8=m_9=0.013379875786.$$

Then there exists a non-empty open set in D_{11} and positive masses M_1, M_2 and M_3 such that the nine bodies with these masses form a central configuration of the 9-body problem.

Considering the point $(x,y)=(0.3,3)\in D_{12}\subset D_1$ and substituting $x=0.3$, $y=3$ into n_1, n_2 and n_3, it follows that $n_1(0.3,3)>0$,

$n_2(0.3,3) > 0, n_3(0.3,3) > 0$. Therefore the components of the mass vector N have the same sign.

In order to give some information about the values of the masses, we consider $M_1 = m_1 = m_2 = m_3 = 1$. Then

$$M_2 = m_4 = m_5 = m_6 = 0.862383986038,$$
$$M_3 = m_7 = m_8 = m_9 = 259.392386239189.$$

Thus there exists a non-empty open set in D_{12} and positive masses M_1, M_2 and M_3 such that the nine bodies with these masses form a central configuration of the 9-body problem. The proof of Theorem 1.1 is completed.

***Remark* 1** Theorem 1.1 can be extended to planar central configurations of the $3n$ -body problem with three nested regular n-gon for $n > 3$.

3 Proof of Theorem 1.2

For the 12-body problem, equations (3) is a system of 66 equations. Without loss of generality, we can assume that

$$m_1(1,0), m_2\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right), m_3\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),$$
$$m_4(-1,0), m_5\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right), m_6\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right),$$
$$m_7(0,x), m_8\left(-\frac{\sqrt{3}}{2}x,-\frac{1}{2}x\right),$$
$$m_9\left(\frac{\sqrt{3}}{2}x,-\frac{1}{2}x\right),$$
$$m_{10}\left(\frac{\sqrt{3}}{2}y,\frac{1}{2}y\right), m_{11}\left(-\frac{\sqrt{3}}{2}y,\frac{1}{2}y\right),$$
$$m_{12}(0,-y) \tag{9}$$

According to Fig. 2, our class of configurations with twelve bodies must satisfy

$$r_{12}=r_{23}=r_{34}=r_{45}=r_{56}=r_{16}=1,$$
$$r_{78}=r_{89}=r_{79}=\sqrt{3}x,$$
$$r_{10,11}=r_{11,12}=r_{10,12}=\sqrt{3}y,$$
$$r_{13}=r_{15}=r_{24}=r_{26}=r_{35}=r_{46}=\sqrt{3},$$
$$r_{14}=r_{25}=r_{36}=2,$$
$$r_{17}=r_{29}=r_{38}=r_{47}=r_{59}=r_{68}=\sqrt{x^2+1},$$
$$r_{18}=r_{28}=r_{39}=r_{49}=r_{57}=r_{67}=\sqrt{x^2+\sqrt{3}x+1},$$
$$r_{19}=r_{27}=r_{37}=r_{48}=r_{58}=r_{69}=\sqrt{x^2-\sqrt{3}x+1},$$
$$r_{1,10}=r_{2,10}=r_{3,11}=r_{4,11}=r_{5,12}=$$
$$r_{6,12}=\sqrt{y^2-\sqrt{3}y+1},$$
$$r_{1,11}=r_{2,12}=r_{3,12}=r_{4,10}=r_{5,10}=$$
$$r_{6,11}=\sqrt{y^2+\sqrt{3}y+1},$$
$$r_{1,12}=r_{2,11}=r_{3,10}=r_{4,12}=r_{5,11}=$$
$$r_{6,10}=\sqrt{y^2+1},$$
$$r_{7,10}=r_{7,11}=r_{8,11}=r_{8,12}=r_{9,12}=$$
$$r_{9,10}=\sqrt{x^2-xy+y^2},$$
$$r_{7,12}=r_{8,10}=r_{9,11}=x+y \tag{10}$$

Taking into account that $M_1 = m_1 = m_2 = m_3 = m_4 = m_5 = m_6, M_2 = m_7 = m_8 = m_9, M_3 = m_{10} = m_{11} = m_{12}$, the equations $f_{12} = 0, f_{23} = 0, f_{34} = 0, f_{45} = 0, f_{56} = 0, f_{16} = 0 f_{78} = 0, f_{79} = 0, f_{89} = 0, f_{14} = 0, f_{25} = 0, f_{36} = 0, f_{7,12} = 0, f_{8,10} = 0, f_{9,11} = 0, f_{10,11} = 0, f_{11,12} = 0$, and $f_{10,12} = 0$ of equations (3) are trivially satisfied. The other 48 equations of equations (3) can be put into 8 sets of equivalent equations:

Set 1. Equations $f_{13} = 0, f_{15} = 0, f_{24} = 0, f_{26} = 0, f_{35} = 0$, and $f_{46} = 0$ can be written as

$$aM_2 - bM_3 = 0 \tag{11}$$

where

$$a = \frac{3}{2}x(R_{18} + R_{19} - 2R_{17}),$$
$$b = \frac{3}{2}y(R_{1,10} + R_{1,11} - 2R_{1,12}).$$

Set 2. Equations $f_{17} = 0, f_{29} = 0, f_{38} = 0, f_{47} = 0, f_{59} = 0$ and $f_{68} = 0$ can be written as

$$b_{11}M_1 + b_{12}M_2 + b_{13}M_3 = 0 \tag{12}$$

where

$$b_{11} = (R_{12} - R_{19})\Delta_{172} + (R_{13} - R_{19})\Delta_{173} + (R_{14} - R_{17})\Delta_{174} + (R_{13} - R_{18})\Delta_{175} + (R_{12} - R_{18})\Delta_{176},$$
$$b_{12} = (R_{18} - R_{78})\Delta_{178} + (R_{19} - R_{78})\Delta_{179},$$
$$b_{13} = (R_{1,10} - R_{7,10})\Delta_{1,7,10} + (R_{1,11} - R_{7,10})\Delta_{1,7,11} + (R_{1,12} - R_{7,12})\Delta_{1,7,12}.$$

Set 3. Equations $f_{18} = 0, f_{28} = 0, f_{39} = 0, f_{49} = 0, f_{57} = 0$ and $f_{67} = 0$.

Set 4. Equations $f_{19} = 0, f_{27} = 0, f_{37} = 0, f_{48} = 0, f_{58} = 0$ and $f_{69} = 0$.

Set 5. Equations $f_{1,10} = 0, f_{2,10} = 0, f_{3,11} = 0, f_{4,11} = 0, f_{5,12} = 0$ and $f_{6,12} = 0$ can be written as

$$b_{21}M_1 + b_{22}M_2 + b_{23}M_3 = 0 \tag{13}$$

where

$$b_{21}=(R_{12}-R_{1,10})\Delta_{1,10,2}+(R_{13}-R_{1,12})\Delta_{1,10,3}+(R_{14}-R_{1,11})\Delta_{1,10,4}+(R_{13}-R_{1,11})\Delta_{1,10,5}+(R_{12}-R_{1,12})\Delta_{1,10,6},$$

$$b_{22}=(R_{17}-R_{7,10})\Delta_{1,10,7}+(R_{18}-R_{7,12})\Delta_{1,10,8}+(R_{19}-R_{7,10})\Delta_{1,10,9},$$

$$b_{23}=(R_{1,11}-R_{10,11})\Delta_{1,10,11}+(R_{1,12}-R_{10,11})\Delta_{1,10,12}.$$

Set 6. Equations $f_{1,11}=0$, $f_{2,12}=0$, $f_{3,12}=0$, $f_{4,10}=0$, $f_{5,10}=0$ and $f_{6,11}=0$.

Set 7. Equations $f_{1,12}=0$, $f_{2,11}=0$, $f_{3,10}=0$, $f_{4,12}=0$, $f_{5,11}=0$ and $f_{6,10}=0$.

Set 8. Equations $f_{7,10}=0$, $f_{7,11}=0$, $f_{8,11}=0$, $f_{8,12}=0$, $f_{9,12}=0$ and $f_{9,10}=0$.

By fundamental calculation, we find that sets 3,4 and sets 6~8 are linear combinations of equations (11), (12) and (13). In order to have planar central configurations according to Fig. 2, we need to find positive solutions M_1, M_2 and M_3 of equations (11)~(13). According to assumptions (9), we have

$$a=\frac{3x}{2}\left[\frac{1}{(x^2-\sqrt{3}x+1)^{\frac{3}{2}}}+\frac{1}{(x^2+\sqrt{3}x+1)^{\frac{3}{2}}}-\frac{2}{(x^2+1)^{\frac{3}{2}}}\right],$$

$$b=\frac{3y}{2}\left[\frac{1}{(y^2-\sqrt{3}y+1)^{\frac{3}{2}}}+\frac{1}{(y^2+\sqrt{3}y+1)^{\frac{3}{2}}}-\frac{2}{(y^2+1)^{\frac{3}{2}}}\right],$$

$$b_{11}=\left(\frac{5}{4}+\frac{1}{\sqrt{3}}\right)x-\frac{2x}{(x^2+1)^{\frac{3}{2}}}-\frac{2x+\sqrt{3}}{(x^2+\sqrt{3}x+1)^{\frac{3}{2}}}-\frac{2x-\sqrt{3}}{(x^2-\sqrt{3}x+1)^{\frac{3}{2}}},$$

$$b_{12}=\frac{\sqrt{3}}{2}x\left[\frac{x+\sqrt{3}}{(x^2+\sqrt{3}x+1)^{\frac{3}{2}}}+\frac{\sqrt{3}-x}{(x^2-\sqrt{3}x+1)^{\frac{3}{2}}}-\frac{2}{3x^3}\right],$$

$$b_{13}=\frac{1}{2}\left[\frac{2x-y-\sqrt{3}xy}{(y^2-\sqrt{3}y+1)^{\frac{3}{2}}}+\frac{2x-y+\sqrt{3}xy}{(y^2+\sqrt{3}y+1)^{\frac{3}{2}}}+\frac{2(x+y)}{(y^2+1)^{\frac{3}{2}}}-\frac{2(2x-y)}{(x^2-xy+y^2)^{\frac{3}{2}}}-\frac{2}{(x+y)^2}\right],$$

$$b_{21}=\frac{1}{2}\left[\left(\frac{5}{4}+\frac{1}{\sqrt{3}}\right)y-\frac{2y}{(y^2+1)^{\frac{3}{2}}}-\frac{2y+\sqrt{3}}{(y^2+\sqrt{3}y+1)^{\frac{3}{2}}}-\frac{2y-\sqrt{3}}{(y^2-\sqrt{3}y+1)^{\frac{3}{2}}}\right],$$

$$b_{22}=\frac{1}{2}\left[\frac{x+y-\sqrt{3}xy}{(x^2-\sqrt{3}x+1)^{\frac{3}{2}}}+\frac{y-2x+\sqrt{3}xy}{(x^2+1)^{\frac{3}{2}}}+\frac{x+y}{(x^2+\sqrt{3}x+1)^{\frac{3}{2}}}-\frac{2y-x}{(x^2-xy+y^2)^{\frac{3}{2}}}-\frac{1}{(x+y)^2}\right],$$

$$b_{23}=\frac{\sqrt{3}}{2}y\left[\frac{\sqrt{3}-y}{(y^2+1)^{\frac{3}{2}}}+\frac{y}{(y^2+\sqrt{3}y+1)^{\frac{3}{2}}}-\frac{1}{3y^3}\right].$$

From (11), we have

$$M_3=\frac{a}{b}M_2 \tag{14}$$

Since the function $f(x)=x^{-\frac{3}{2}}$ is convex in the interval $(0,+\infty)$, so the coefficients a and b are positive in the interval $(0,+\infty)$. In order to have positive solutions M_1, M_2 and M_3 of equations (11)~(13), the coefficients a and b must have same sign, so we have $0<x<y<1$, $0<x<1$ and $y>1$ or $1<x<y$, that is, the point (x,y) is in the region D_2 (see Fig. 6). Substituting (14) into equations (12) and (13), we have

$$bb_{11}M_1+(bb_{12}+ab_{13})M_2=0 \tag{15}$$

$$bb_{21}M_1+(bb_{22}+ab_{23})M_2=0 \tag{16}$$

Therefore, there will be positive masses M_1 and M_2 solutions of equations (15) and (16) if and only if the determinant

$$A_1=\begin{vmatrix} bb_{11} & bb_{12}+ab_{13} \\ bb_{21} & bb_{22}+ab_{23} \end{vmatrix}$$

is zero and $A_2=bb_{11}(bb_{12}+ab_{13})$ is negative. Considering a straight line $y=4x$ in the region D_2, the straight line $y=4x$ and the curve $A_1=0$ have three intersection P_1, P_2 and P_3 in the region D_{21}, D_{22} and D_{23}, respectively.

Consider the point $P_1 = (0.218924200485, 0.875696801940) \in D_{21} \subset D_2$, it follows that $A_2 < 0$.

In order to give some information about the values of the masses, we consider $M_1 = m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1$. Then

$M_2 = m_7 = m_8 = m_9 = 0.095628911130$,

$M_3 = m_{10} = m_{11} = m_{12} = 0.018007796440$.

Thus there exists a non-empty open set in D_{21} and positive masses M_1, M_2 and M_3 such that the twelve bodies with these masses form a central configuration of the 12-body problem.

Consider the point $P_2 = (0.826973538630, 3.307894154520) \in D_{22} \subset D_2$, it follows that $A_2 < 0$. In order to give some information about the values of the masses, we consider $M_1 = m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1$. Then

$M_2 = m_7 = m_8 = m_9 = 2.292478218333$,

$M_3 = m_{10} = m_{11} = m_{12} = 139.402747556039$.

Thus there exists a non-empty open set in D_{22} and positive masses M_1, M_2 and M_3 such that the twelve bodies with these masses form a central configuration of the 12-body problem.

Consider the point $P_3 = (2.439066222221, 9.756264888884) \in D_{23} \subset D_2$, it follows that $A_2 < 0$. In order to give some information about the values of the masses, we consider $M_1 = m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1$. Then

$M_2 = m_7 = m_8 = m_9 = 2.488927597396$,

$M_3 = m_{10} = m_{11} = m_{12} = 681.137415705532$.

Thus there exists a non-empty open set in D_{23} and positive masses M_1, M_2 and M_3 such that the twelve bodies with these masses form a central configuration of the 12-body problem. Hence the proof of Theorem 1.2 is completed.

***Remark* 2** There does not exist central configuration of planar 9-body problem if we rotate one of triangle with an angle of $\pi/3$ in Theorem 1.2.

4 Proof of Theorem 1.3

For the 12-body problem, equations (3) is a system of 66 equations. Without loss of generality, we can assume that

$$m_1(1,0), m_2\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right), m_3\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),$$
$$m_4(-1,0), m_5\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right),$$
$$m_6\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right), m_7(x,0),$$
$$m_8\left(-\frac{1}{2}x,\frac{\sqrt{3}}{2}x\right), m_9\left(-\frac{1}{2}x,-\frac{\sqrt{3}}{2}x\right),$$
$$m_{10}(y,0), m_{11}\left(-\frac{1}{2}y,\frac{\sqrt{3}}{2}y\right),$$
$$m_{12}\left(-\frac{1}{2}y,-\frac{\sqrt{3}}{2}y\right) \tag{17}$$

According to Fig. 3, our class of configurations with twelve bodies must satisfy

$$\begin{aligned}
&r_{12}=r_{23}=r_{34}=r_{45}=r_{56}=r_{16}=1,\\
&r_{78}=r_{89}=r_{79}=\sqrt{3}x,\\
&r_{10,11}=r_{11,12}=r_{10,12}=\sqrt{3}y,\\
&r_{13}=r_{15}=r_{24}=r_{26}=r_{35}=r_{46}=\sqrt{3},\\
&r_{14}=r_{25}=r_{36}=2,\\
&r_{17}=r_{38}=r_{59}=\sqrt{(x-1)^2},\\
&r_{18}=r_{19}=r_{37}=r_{39}=r_{57}=r_{58}=\sqrt{x^2+x+1},\\
&r_{27}=r_{28}=r_{48}=r_{49}=r_{67}=r_{69}=\\
&\quad\sqrt{x^2-x+1} \qquad (18)\\
&r_{29}=r_{47}=r_{68}=x+1,\\
&r_{1,10}=r_{3,11}=r_{5,12}=\sqrt{(y-1)^2},\\
&r_{1,11}=r_{1,12}=r_{3,10}=r_{3,12}=r_{5,10}=\\
&\quad r_{5,11}=\sqrt{y^2+y+1},\\
&r_{2,10}=r_{2,11}=r_{4,11}=r_{4,12}=r_{6,10}=\\
&\quad r_{6,12}=\sqrt{y^2-y+1},\\
&r_{2,12}=r_{4,10}=r_{6,11}=y+1,\\
&r_{7,10}=r_{8,11}=r_{9,12}=\sqrt{(y-x)^2},\\
&r_{7,11}=r_{7,12}=r_{8,10}=r_{8,12}=r_{9,10}=\\
&\quad r_{9,11}=\sqrt{x^2+xy+y^2}.
\end{aligned}$$

Taking into account that $M_1 = m_1 = m_2 = m_3 = m_4 = m_5 = m_6, M_2 = m_7 = m_8 = m_9, M_3 = m_{10} = m_{11} = m_{12}$, the equations $f_{13} = 0, f_{15} = 0, f_{24} = 0, f_{26} = 0, f_{35} = 0, f_{46} = 0, f_{78} = 0, f_{79} = 0, f_{89} = 0, f_{14} = 0, f_{25} = 0, f_{36} = 0, f_{17} = 0, f_{38} = 0, f_{59} = 0, f_{29} = 0, f_{47} = 0, f_{68} = 0, f_{1,10} = 0, f_{3,11} = 0, f_{5,12} = 0, f_{2,12} = 0, f_{4,10} = 0, f_{6,11} = 0, f_{7,10} = 0, f_{8,11} = 0, f_{9,12} = 0, f_{10,11} = 0, f_{11,12} = 0$ and $f_{10,12} = 0$ of equations (3) are trivially satisfied. The other 36 equations of equations (3) can be put into 6 sets of equivalent equations:

Set 1. Equations $f_{12}=0, f_{23}=0, f_{34}=0, f_{45}=0, f_{56}=0$ and $f_{16}=0$ can be written as

$$\alpha M_2+\beta M_3=0 \tag{19}$$

where

$$\alpha=(R_{17}-R_{27})\Delta_{127}+(R_{18}-R_{27})\Delta_{128}+(R_{18}-R_{29})\Delta_{129},$$
$$\beta=(R_{1,10}-R_{2,10})\Delta_{1,2,10}+(R_{1,11}-R_{2,10})\Delta_{1,2,11}+(R_{1,11}-R_{2,12})\Delta_{1,2,12}.$$

Set 2. Equations $f_{18}=0, f_{19}=0, f_{37}=0, f_{39}=0, f_{57}=0$ and $f_{58}=0$ can be written as

$$c_{11}M_1+c_{12}M_2+c_{13}M_3=0 \tag{20}$$

where

$$c_{11}=(R_{12}-R_{27})\Delta_{182}+(R_{13}-R_{17})\Delta_{183}+(R_{14}-R_{27})\Delta_{184}+(R_{13}-R_{18})\Delta_{185}+(R_{12}-R_{29})\Delta_{186},$$
$$c_{12}=(R_{17}-R_{78})\Delta_{187}+(R_{18}-R_{78})\Delta_{189},$$
$$c_{13}=(R_{1,10}-R_{7,11})\Delta_{1,8,10}+(R_{1,11}-R_{7,10})\Delta_{1,8,11}+(R_{1,11}-R_{7,11})\Delta_{1,8,12}.$$

Set 3. Equations $f_{27}=0, f_{28}=0, f_{48}=0, f_{49}=0, f_{67}=0$ and $f_{69}=0$.

Set 4. Equations $f_{1,11}=0, f_{1,12}=0, f_{3,10}=0, f_{3,12}=0, f_{5,10}=0$ and $f_{5,11}=0$ can be written as

$$c_{21}M_1+c_{22}M_2+c_{23}M_3=0 \tag{21}$$

where

$$c_{21}=(R_{12}-R_{2,10})\Delta_{1,11,2}+(R_{13}-R_{1,10})\Delta_{1,11,3}+(R_{14}-R_{2,10})\Delta_{1,11,4}+(R_{13}-R_{1,11})\Delta_{1,11,5}+(R_{12}-R_{2,12})\Delta_{1,11,6},$$
$$c_{22}=(R_{17}-R_{7,11})\Delta_{1,11,7}+(R_{18}-R_{7,10})\Delta_{1,11,8}+(R_{18}-R_{7,11})\Delta_{1,11,9},$$
$$c_{23}=(R_{1,10}-R_{10,11})\Delta_{1,11,10}+(R_{1,11}-R_{10,11})\Delta_{1,11,12}.$$

Set 5. Equations $f_{2,10}=0, f_{2,11}=0, f_{4,11}=0, f_{4,12}=0, f_{6,10}=0$ and $f_{6,12}=0$.

Set 6. Equations $f_{7,11}=0, f_{7,12}=0, f_{8,10}=0, f_{8,12}=0, f_{9,10}=0$ and $f_{9,11}=0$.

By fundamental calculation, we find that set 3 and sets 5, 6 are linear combinations of equations (19)~(21). In order to have planar central configurations according to Fig. 3, we need to find positive solutions M_1, M_2 and M_3 of equations (19)~(21). According to assumptions (17), we have

$$\alpha=\frac{\sqrt{3}}{2}\left[\frac{1-x}{(x^2-2x+1)^{\frac{3}{2}}}+\frac{2+x}{(x^2+x+1)^{\frac{3}{2}}}-\frac{2-x}{(x^2-x+1)^{\frac{3}{2}}}-\frac{1}{(x+1)^2}\right],$$

$$\beta=\frac{\sqrt{3}}{2}\left[\frac{1-y}{(y^2-2y+1)^{\frac{3}{2}}}+\frac{2+y}{(y^2+y+1)^{\frac{3}{2}}}-\frac{2-y}{(y^2-y+1)^{\frac{3}{2}}}-\frac{1}{(y+1)^2}\right],$$

$$c_{11}=\left(\frac{5}{4}+\frac{1}{\sqrt{3}}\right)x-\frac{x-1}{(x^2-2x+1)^{\frac{3}{2}}}-\frac{2x+1}{(x^2+x+1)^{\frac{3}{2}}}-\frac{2x-1}{(x^2-x+1)^{\frac{3}{2}}}-\frac{1}{(x+1)^2}$$

$$c_{12}=\frac{x(x+2)}{(x^2+x+1)^{\frac{3}{2}}}+\frac{x(1-x)}{(x^2-2x+1)^{\frac{3}{2}}}-\frac{1}{\sqrt{3}x^2},$$

$$c_{13}=\frac{x(1-y)}{(y^2-2y+1)^{\frac{3}{2}}}+\frac{x(2+y)}{(y^2+y+1)^{\frac{3}{2}}}-\frac{2x+y}{(x^2+xy+y^2)^{\frac{3}{2}}}-\frac{x-y}{(x^2-2xy+y^2)^{\frac{3}{2}}},$$

$$c_{21}=\left(\frac{5}{4}+\frac{1}{\sqrt{3}}\right)y-\frac{y-1}{(y^2-2y+1)^{\frac{3}{2}}}-\frac{2y+1}{(y^2+y+1)^{\frac{3}{2}}}-\frac{2y-1}{(y^2-y+1)^{\frac{3}{2}}}-\frac{1}{(y+1)^2},$$

$$c_{22}=\frac{y(1-x)}{(x^2-2x+1)^{\frac{3}{2}}}+\frac{y(2+x)}{(x^2+x+1)^{\frac{3}{2}}}-\frac{2y+x}{(x^2+xy+y^2)^{\frac{3}{2}}}-\frac{y-x}{(x^2-2xy+y^2)^{\frac{3}{2}}},$$

$$c_{23}=\frac{y(y+2)}{(y^2+y+1)^{\frac{3}{2}}}+\frac{y(1-y)}{(y^2-2y+1)^{\frac{3}{2}}}-\frac{1}{\sqrt{3}y^2}.$$

From (19), we have

$$M_3=-\frac{\alpha}{\beta}M_2 \tag{22}$$

The function

$$g(x)=\frac{1-x}{(x^2-2x+1)^{\frac{3}{2}}}+\frac{2+x}{(x^2+x+1)^{\frac{3}{2}}}-\frac{2-x}{(x^2-x+1)^{\frac{3}{2}}}-\frac{1}{(x+1)^2}$$

is positive in the interval (0,1) and is negative in the interval $(1, +\infty)$, so the coefficients α and β are positive in the interval $(0,1)$ and are negative in the interval $(1, +\infty)$. In order to have positive solutions M_1, M_2 and M_3 of equations (19)~(21), the coefficients α and β must have opposite sign, so it follows $0 < x < 1$ and $y > 1$, that is, the point (x, y) is in the region D_3 (see Fig. 7). Substituting (22) into equations (20) and (21), we have

$$\beta c_{11} M_1 + (\beta c_{12} - \alpha c_{13}) M_2 = 0 \tag{23}$$

$$\beta c_{21} M_1 + (\beta c_{22} - \alpha c_{23}) M_2 = 0 \tag{24}$$

Therefore there will be positive masses M_1 and M_2 solutions of equations (23) and (24) if and only if the determinant

$$B_1 = \begin{vmatrix} \beta c_{11} & \beta c_{12} - \alpha c_{13} \\ \beta c_{21} & \beta c_{22} - \alpha c_{23} \end{vmatrix}$$

is zero and $B_2 = \beta c_{11}(\beta c_{12} - \alpha c_{13})$ is negative. Considering a straight line $y = 2x + 1$ in the region D_3, the straight line $y = 2x + 1$ and the curve $B_1 = 0$ have a intersection Q in the region D_3.

Consider the point $Q = (0.302606931501, 1.605213863002) \in D_3$, it follows that $B_2 < 0$. In order to give some informations about the values of the masses, we consider $M_1 = m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1$. Then

$$M_2 = m_7 = m_8 = m_9 = 0.300691732891,$$

$$M_3 = m_{10} = m_{11} = m_{12} = 0.049115765210.$$

Thus there exists a non-empty open set in D_3 and positive masses M_1, M_2 and M_3 such that the twelve bodies with these masses form a central configuration of the 12-body problem. In short we have proved Theorem 1.3.

5 Proof of Theorem 1.4

For the 12-body problem, equations (3) is a system of 66 equations. Without loss of generality, we can assume that

$$m_1(1,0), m_2\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), m_3\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

$$m_4(-1,0), m_5\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),$$

$$m_6\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), m_7(x,0),$$

$$m_8\left(-\frac{1}{2}x, \frac{\sqrt{3}}{2}x\right), m_9\left(-\frac{1}{2}x, -\frac{\sqrt{3}}{2}x\right),$$

$$m_{10}\left(\frac{1}{2}y, \frac{\sqrt{3}}{2}y\right), m_{11}(-y,0),$$

$$m_{12}\left(\frac{1}{2}y, -\frac{\sqrt{3}}{2}y\right) \tag{25}$$

According to Fig. 4, our class of configurations with twelve bodies must satisfy

$$r_{12} = r_{23} = r_{34} = r_{45} = r_{56} = r_{16} = 1,$$
$$r_{78} = r_{89} = r_{79} = \sqrt{3}x,$$
$$r_{10,11} = r_{11,12} = r_{10,12} = \sqrt{3}y,$$
$$r_{13} = r_{15} = r_{24} = r_{26} = r_{35} = r_{46} = \sqrt{3},$$
$$r_{14} = r_{25} = r_{36} = 2, r_{17} = r_{38} = r_{59} = \sqrt{(x-1)^2},$$
$$r_{18} = r_{19} = r_{37} = r_{39} = r_{57} = r_{58} = \sqrt{x^2 + x + 1},$$
$$r_{27} = r_{28} = r_{48} = r_{49} = r_{67} = r_{69} = \sqrt{x^2 - x + 1},$$
$$r_{29} = r_{47} = r_{68} = x + 1, r_{1,11} = r_{3,12} = r_{5,10} = y + 1,$$
$$r_{1,10} = r_{1,12} = r_{3,10} = r_{3,11} = r_{5,11} =$$
$$r_{5,12} = \sqrt{y^2 - y + 1},$$
$$r_{2,11} = r_{2,12} = r_{4,10} = r_{4,12} = r_{6,10} =$$
$$r_{6,11} = \sqrt{y^2 + y + 1},$$
$$r_{2,10} = r_{4,11} = r_{6,12} = \sqrt{(y-1)^2},$$
$$r_{7,11} = r_{8,12} = r_{9,10} = x + y,$$
$$r_{7,10} = r_{7,12} = r_{8,10} = r_{8,11} = r_{9,11} = r_{9,12} =$$
$$\sqrt{x^2 - xy + y^2} \tag{26}$$

Taking into account that $M_1 = m_1 = m_2 = m_3 = m_4 = m_5 = m_6$, $M_2 = m_7 = m_8 = m_9$, $M_3 = m_{10} = m_{11} = m_{12}$, the equations $f_{13} = 0, f_{15} = 0, f_{24} = 0, f_{26} = 0, f_{35} = 0, f_{46} = 0, f_{78} = 0, f_{79} = 0, f_{89} = 0, f_{14} = 0, f_{25} = 0, f_{36} = 0, f_{17} = 0, f_{38} = 0, f_{59} = 0, f_{29} = 0, f_{47} = 0, f_{68} = 0, f_{1,11} = 0, f_{3,12} = 0, f_{5,10} = 0, f_{2,10} = 0, f_{4,11} = 0, f_{6,12} = 0, f_{7,11} = 0, f_{8,12} = 0, f_{9,10} = 0, f_{10,11} = 0, f_{11,12} = 0$ and $f_{10,12} = 0$ of equations (3) are trivially satisfied. The other 36 equations of equations (3) can be put into 6 sets of equivalent equations:

Set 1. Equations $f_{12} = 0, f_{23} = 0, f_{34} = 0, f_{45} = 0, f_{56} = 0$ and $f_{16} = 0$ can be written as

$$\alpha M_2 - \beta M_3 = 0 \tag{27}$$

where

$$\alpha = (R_{17} - R_{27})\Delta_{127} + (R_{18} - R_{27})\Delta_{128} + (R_{18} - R_{29})\Delta_{129},$$

$$\beta = (R_{1,10} - R_{2,10})\Delta_{1,2,10} + (R_{1,11} - R_{2,11})\Delta_{1,2,11} + (R_{1,10} - R_{2,11})\Delta_{1,2,12}.$$

Set 2. Equations $f_{18}=0, f_{19}=0, f_{37}=0, f_{39}=0, f_{57}=0$ and $f_{58}=0$ can be written as

$$d_{11}M_1+d_{12}M_2+d_{13}M_3=0 \tag{28}$$

where

$$d_{11}=(R_{12}-R_{27})\Delta_{182}+(R_{13}-R_{17})\Delta_{183}+(R_{14}-R_{27})\Delta_{184}+(R_{13}-R_{18})\Delta_{185}+(R_{12}-R_{29})\Delta_{186},$$
$$d_{12}=(R_{17}-R_{78})\Delta_{187}+(R_{18}-R_{78})\Delta_{189},$$
$$d_{13}=(R_{1,10}-R_{7,10})\Delta_{1,8,10}+(R_{1,11}-R_{7,10})\Delta_{1,8,11}+(R_{1,10}-R_{7,11})\Delta_{1,8,12}.$$

Set 3. Equations $f_{27}=0, f_{28}=0, f_{48}=0, f_{49}=0, f_{67}=0$ and $f_{69}=0$.

Set 4. Equations $f_{1,10}=0, f_{1,12}=0, f_{3,10}=0, f_{3,11}=0, f_{5,11}=0$ and $f_{5,12}=0$ can be written as

$$d_{21}M_1+d_{22}M_2+d_{23}M_3=0 \tag{29}$$

where

$$d_{21}=(R_{12}-R_{2,10})\Delta_{1,10,2}+(R_{13}-R_{1,10})\Delta_{1,10,3}+(R_{14}-R_{2,11})\Delta_{1,10,4}+(R_{13}-R_{1,11})\Delta_{1,10,5}+(R_{12}-R_{2,11})\Delta_{1,10,6},$$
$$d_{22}=(R_{17}-R_{7,10})\Delta_{1,10,7}+(R_{18}-R_{7,10})\Delta_{1,10,8}+(R_{18}-R_{7,11})\Delta_{1,10,9},$$
$$d_{23}=(R_{1,11}-R_{10,11})\Delta_{1,10,11}+(R_{1,10}-R_{10,11})\Delta_{1,10,12}.$$

Set 5. Equations $f_{2,11}=0, f_{2,12}=0, f_{4,10}=0, f_{4,12}=0, f_{6,10}=0$ and $f_{6,11}=0$.

Set 6. Equations $f_{7,10}=0, f_{7,12}=0, f_{8,10}=0, f_{8,11}=0, f_{9,11}=0$ and $f_{9,12}=0$.

By fundamental calculation, we find that set 3 and sets 5,6 are linear combinations of equations (27)~(29). In order to have planar central configurations according to Fig. 3, we need to find positive solutions M_1, M_2 and M_3 of equations (27)~(29). According to assumptions (25), we have

$$d_{11}=\left(\frac{5}{4}+\frac{1}{\sqrt{3}}\right)x-\frac{x-1}{(x^2-2x+1)^{\frac{3}{2}}}-\frac{2x+1}{(x^2+x+1)^{\frac{3}{2}}}-\frac{2x-1}{(x^2-x+1)^{\frac{3}{2}}}-\frac{1}{(x+1)^2},$$

$$d_{12}=\frac{x(x+2)}{(x^2+x+1)^{\frac{3}{2}}}+\frac{x(1-x)}{(x^2-2x+1)^{\frac{3}{2}}}-\frac{1}{\sqrt{3}x^2},$$

$$d_{13}=\frac{x(2-y)}{(y^2-y+1)^{\frac{3}{2}}}+\frac{x}{(y+1)^2}-\frac{2x-y}{(x^2-xy+y^2)^{\frac{3}{2}}}-\frac{1}{(x+y)^2},$$

$$d_{21}=\left(\frac{5}{4}+\frac{1}{\sqrt{3}}\right)y-\frac{y-1}{(y^2-2y+1)^{\frac{3}{2}}}-\frac{2y+1}{(y^2+y+1)^{\frac{3}{2}}}-\frac{2y-1}{(y^2-y+1)^{\frac{3}{2}}}-\frac{1}{(y+1)^2},$$

$$d_{22}=\frac{y(1-x)}{(x^2-2x+1)^{\frac{3}{2}}}+\frac{y(2+x)}{(x^2+x+1)^{\frac{3}{2}}}-\frac{2y-x}{(x^2-xy+y^2)^{\frac{3}{2}}}-\frac{1}{(x+y)^2},$$

$$c_{23}=\frac{y(2-y)}{(y^2-y+1)^{\frac{3}{2}}}+\frac{y}{(y+1)^2}-\frac{1}{\sqrt{3}y^2}.$$

From (27), we have

$$M_3=\frac{\alpha}{\beta}M_2 \tag{30}$$

In order to have positive solutions M_1, M_2 and M_3 of equations (27)~(29), the coefficients α and β must have same sign, so it follows $0<x<y<1$ and $1<x<y$, that is, the point (x,y) is in the region D_4 (see Fig. 8). Substituting (30) into equations (28) and (29), we have

$$\beta d_{11}M_1+(\beta d_{12}+\alpha d_{13})M_2=0 \tag{31}$$

$$\beta d_{21}M_1+(\beta d_{22}+\alpha d_{23})M_2=0 \tag{32}$$

Therefore there will be positive masses M_1 and M_2 solutions of equations (31) and (32) if and only if the determinant

$$C_1=\begin{vmatrix}\beta d_{11} & \beta d_{12}+\alpha d_{13}\\ \beta d_{21} & \beta d_{22}+\alpha d_{23}\end{vmatrix}$$

is zero and $C_2=\beta d_{11}(\beta d_{12}+\alpha d_{13})$ is negative. Considering a straight line $y=\frac{3}{2}x$ in the region D_{41}, the straight line $y=\frac{3}{2}x$ and the curve $C_1=0$ have a intersection K_1 in the region D_{41}.

Consider the point $K_1=(0.485133252406, 0.727699878609)\in D_{41}$, it follows that $C_2<0$. In order to give some informations about the values of the masses, we consider $M_1=m_1=m_2=m_3=m_4=m_5=m_6=1$. Then

$$M_2=m_7=m_8=m_9=18.023447667221,$$
$$M_3=m_{10}=m_{11}=m_{12}=3.090055655630.$$

Thus there exists a non-empty open set in D_{41} and positive masses M_1, M_2 and M_3 such that the twelve bodies with these masses form a central configuration of the 12-body problem.

Considering a straight line $y = 4x$ in the region D_{42}, the straight line $y = 4x$ and the curve $C_1 = 0$ have a intersection K_2 in the region D_{42}.

Consider the point $K_2 = (2.479104019882, 9.916416079529) \in D_{42}$, it follows that $C_2 < 0$. In order to give some informations about the values of the masses, we consider $M_1 = m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1$. Then

$$M_2 = m_7 = m_8 = m_9 = 2.374630964979,$$

$$M_3 = m_{10} = m_{11} = m_{12} = 693.243523466378.$$

Thus there exists a non-empty open set in D_{42} and positive masses M_1, M_2 and M_3 such that the twelve bodies with these masses form a central configuration of the 12-body problem. In short we have proved Theorem 1.4.

***Remark* 3** Theorem 1.2～1.4 can be extended to planar central configurations of the $n+n+2n$-body problem for $n>4$.

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