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加权 Bergman-Orlicz 空间到有界型空间上的加权迭代径向算子

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摘　要：设 $B^n=\{z\in\mathbf{C}^n:|z|<1\}$ 是 n 维复平面 $\mathbf{C}^n$ 中的开单位球，$H(B^n)$ 是 B^n 上的全纯函数集合. 设 $u\in H(B^n)$，$m\in\mathbf{N}$. 本文通过在加权 Bergman-Orlicz 空间中构造合适的测试函数，利用符号函数 u 刻画了加权 Bergman-Orlicz 空间到有界型空间上的加权迭代径向算子 $\mathfrak{R}_u^m$ 的有界性和紧致性.

关键词：加权 Bergman-Orlicz 空间；有界型空间；加权迭代径向算子；有界性；紧致性
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Weighted iterated radial operators from weighted Bergman-Orlicz spaces to bounded-type spaces

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Abstract: Let B^n be the open unit disk in $\mathbf{C}^n$ and $H(B^n)$ the class of all analytic functions on B^n. Let $u\in H(B^n)$ and $m\in\mathbf{N}$. The boundedness and compactness of weighted iterated radial operators from weighted Bergman-Orlicz spaces to a class of bounded-type spaces are characterized by constructing some special functions.

Keywords: Weighted Bergman-Orlicz space; Bounded-type space; Weighted iterated radial operator; Boundedess; Compactness
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1 Introduction

Let $B^n=\{z\in\mathbf{C}^n:|z|<1\}$ the unit ball in $\mathbf{C}^n$ and $H(B^n)$ the class of all holomorphic functions on B^n. Let R be the radial derivative operator on some subspaces of $H(B^n)$, that is,

$$\mathfrak{R}f(z)=\sum_{j=1}^{n}z_j\frac{\partial f}{\partial z_j}(z).$$

If we consider the Taylor expansion $f(z)=\sum_{|\beta|\geq 0}a_\beta z^\beta$, then

$$\mathfrak{R}f(z)=\sum_{|\beta|\geq 0}|\beta|a_\beta z^\beta,$$

where $\beta=(\beta_1,\beta_2,\cdots,\beta_n)$ is a multi-index, $|\beta|=\beta_1+\beta_2+\cdots+\beta_n$ and $z^\beta=z^{\beta_1}\cdots z^{\beta_n}$.

The iterated radial derivative operator $\mathfrak{R}^m f$ on some subspaces of $H(B^n)$ is defined inductively by $\mathfrak{R}^m f=\mathfrak{R}(\mathfrak{R}^{m-1}f)$, $m\in\mathbf{N}\setminus\{1\}$. We regard that $\mathfrak{R}^0$ is the identity operator, that is, $\mathfrak{R}^0 f=f$ for every $f\in H(B^n)$.

Let $u\in H(B^n)$ The weighted iterated radial operator on some subspaces of $H(B^n)$ is defined

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by $\mathfrak{R}_u^m f = u(\mathfrak{R}^m f), m \in N\backslash\{1\}$.

It is interesting to consider when u induces a bounded or compact operator $\mathfrak{R}_u^m$ on or between some subspaces of $H(B^n)$[1-10]. This kind of operator was introduced and studied by Stevic in Ref. [1]. Quite recently, products of radial derivative and weighted composition operators from weighted Bergman-Orlicz spaces to weighted-type spaces have been studied in Ref. [2]. As a continuation of the investigation of concrete operators, here we study the operator $\mathfrak{R}_u^m$ from weighted Bergman-Orlicz spaces to weighted-type spaces.

2 Preliminaries

Let dv be the Lebesgue measure on B^n, $d\sigma$ the normalized surface measure on $S_n = \partial B^n$ (the boundary of B^n). Let $z=(z_1, z_2, \cdots, z_n)$ and $w=(w_1, w_2, \cdots, w_n)$ be points in $\mathbf{C}^n$, $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ and $|z|^2 = \langle z, z\rangle$. For $\alpha > -1$, by dv_α we denote the normalized Lebesgue measure $c_\alpha(1-|z|^2)^\alpha dv(z)$ (constant c_α is chosen such that $v_\alpha(B^n)=1$).

The following facts come from Ref. [3]. The function $\Phi \neq 0$ is called a growth function, if it is a continuous and non-decreasing function from the interval $[0, \infty)$ onto itself. Clearly, these conditions, among others, imply that $\Phi(0)=0$. The function Φ is of positive upper type $q \geq 1$, if there exists $C > 0$ such that $\Phi(st) \leqslant Ct^q\Phi(s)$ for every $s > 0$ and $t \geq 1$. We denote by $\wp^q$ the set of growth functions Φ of positive upper type q (for some $q \geq 1$), such that the function $t \to \dfrac{\Phi(t)}{t}$ is non-decreasing on $(0, \infty)$. The function Φ is of positive lower type $p > 0$, if there exists $C > 0$ such that $\Phi(st) \leqslant Ct^p\Phi(s)$ for every $s > 0$ and $0 < t \leqslant 1$. By $\mathfrak{I}_p$ we denote the set of growth functions Φ of positive lower type p (for some $0 < p \leqslant 1$), such that the function $t \to \dfrac{\Phi(t)}{t}$ is non-increasing on $(0, \infty)$.

Let Φ be a growth function. The weighted Bergman-Orlicz space $A_\alpha^\Phi(B^n)$ consists of all $f \in H(B^n)$ such that

$$\| f \|_{A_\alpha^\Phi(B^n)} = \int_{B_n} \Phi(|f(z)|) dv_\alpha(z) < \infty.$$

On $A_\alpha^\Phi(B^n)$ is defined the following quasi-norm

$$\| f \|_{A_\alpha^\Phi(B^n)}^{lux} = \inf\left\{\lambda > 0: \int_{B_n} \Phi\left(\frac{|f(z)|}{\lambda}\right) dv_\alpha(z) < 1\right\}.$$

If $\Phi \in \wp^q$ or $\Phi \in \mathfrak{I}_p$, then the quasi-norm on $A_\alpha^\Phi(B^n)$ is finite and call the Luxembourg norm.

The classical weighted Bergman space $A_\alpha^p(B^n)$, $p>0$, $\alpha>-1$, corresponds to $\Phi(t)=t^p$ and consists of all $f \in H(B^n)$ such that

$$\| f \|_{A_\alpha^p(B^n)}^p = \int_{B_n} |f(z)|^p dv_\alpha(z) < \infty.$$

We say that a function $\omega: (0,1] \to (0, \infty)$ belongs to class Ω_1, if ω is non-increasing, $\dfrac{1}{\omega}$ is of some positive lower type and the function $t\omega$ is increasing. For example, the function $\omega(t) = \dfrac{1}{t^\alpha}$, $0 < \alpha < 1$, belongs to class Ω_1. We say that a function $\omega: (0,1] \to (0, \infty)$ belongs to class Ω_2, if $\omega \in \mathfrak{I}_p$ and satisfies the condition

$$\int_t^1 \frac{\omega(s)}{s^2} ds \prec \frac{\omega(t)}{t}, (0 < t < 1).$$

Let ω be a positive function defined on $(0,1]$. An $f \in H(B^n)$ is said to be in $H_\omega^\infty(B^n)$, if

$$\| f \|_{H_\omega^\infty(B^n)} = \sup_{z \in B^n} \frac{|f(z)|}{\omega(1-|z|)} < \infty.$$

It is easy to see that $H_\omega^\infty(B^n)$ is a Banach space with the norm $\| \cdot \|_{H_\omega^\infty(B^n)}$. The space $H_\omega^\infty(B^n)$ with $\omega \in \Omega_1$ is not quite often used in the literature. It seems to first appear in Ref. [4] as far as we know.

Let X and Y be topological vector spaces whose topologies are given by translation invariant metrics dX and dY, respectively. Let $L: X \to Y$ be a linear operator. The operator $L: X \to Y$ is bounded if there exists a positive constant K such that $d_Y(Lf, 0) \leqslant K d_X(f, 0)$ for all $f \in X$. The operator $L: X \to Y$ is compact if it maps bounded sets into relatively compact sets.

Throughout this paper, positive constant C may differ from one occurrence to the other. The notation $a \prec b$ means that $a \leqslant Cb$ for some positive constant C.

We first have the following compactness criteria. Since the proof is similar to that of Proposition 3.11 in Ref. [5], it is omitted.

Lemma 2.1 The operator $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$ (or) is compact if and only if for every bounded sequence $\{f_j\}$ in $A_\alpha^\Phi(B^n)$ such that $f_j\to 0$ uniformly on any compact subset of B^n as $j\to\infty$, it follows that $\lim_{j\to\infty}\|\mathfrak{R}_u^m f_j\|_{H_\omega^\infty(B^n)}=0$.

We need the following estimate. For the cases of $m=0$ and $m=1$, they were obtained in Refs. [8] and [9], respectively.

Lemma 2.2 Let $\Phi\in\mathfrak{S}^q\cup\mathfrak{J}_p$ and $m\in\mathbf{N}$. Then there exist two positive constants C and D independent of $f\in A_\alpha^\Phi(B^n)$ and $z\in B^n$ such that

$$|\mathfrak{R}^m f(z)|\leqslant \frac{C}{(1-|z|^2)^m}\Phi^{-1}\left(\frac{D}{(1-|z|^2)^{n+1+\alpha}}\right)\|f\|_{A_\alpha^\Phi(B^n)}^{lux}.$$

Proof First we consider the case where $\Phi\in\mathfrak{S}^q$. We observe that in this case, $A_\alpha^\Phi(B^n)$ continuously embeds into $A_\alpha^1(B^n)$. Then for any $f\in A_\alpha^\Phi(B^n)$ and $z\in B^n$, we have

$$f(z)=\int_{B^n}\frac{f(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}dv_\alpha(w) \qquad (1)$$

For any $\beta=(\beta_1,\beta_2,\cdots,\beta_n)$, we write $|\beta|=N$. From (1), it follows that

$$\frac{\partial^N f}{\partial z^\beta}(z)=c\int_{B^n}\frac{\bar{w}^\beta f(w)}{(1-\langle z,w\rangle)^{n+1+N+\alpha}}dv_\alpha(w) \qquad (2)$$

From (2), we have

$$\frac{(1-|z|^2)^N}{\|f\|_{A_\alpha^\Phi(B^n)}^{lux}}\left|\frac{\partial^N f}{\partial z^\beta}(z)\right|\leqslant c\int_{B^n}\frac{|f(w)|}{\|f\|_{A_\alpha^\Phi(B^n)}^{lux}}\frac{(1-|z|^2)^N}{|1-\langle z,w\rangle|^{n+1+N+\alpha}}dv_\alpha(w) \qquad (3)$$

Using Proposition1.4.10 in Ref. [8], it is easy to see that $\frac{(1-|z|^2)^N}{|1-\langle z,w\rangle|^{n+1+N+\alpha}}dv_\alpha(w)$ is up to a constant a probability measure. Hence from the convexity and Jensen's inequality, we obtain

$$\Phi\left(\frac{(1-|z|^2)^N}{\|f\|_{A_\alpha^\Phi(B^n)}^{lux}}\left|\frac{\partial^N f}{\partial z^\beta}(z)\right|\right)\leqslant C\int_{B^n}\Phi\left(\frac{|f(w)|}{\|f\|_{A_\alpha^\Phi(B^n)}^{lux}}\right)\cdot\frac{(1-|z|^2)^N}{|1-\langle z,w\rangle|^{n+1+N+\alpha}}dv_\alpha(w)\leqslant \frac{C}{(1-|z|^2)^{n+1+\alpha}}\int_{B^n}\Phi\left(\frac{|f(w)|}{\|f\|_{A_\alpha^\Phi(B^n)}^{lux}}\right)dv_\alpha(w)\leqslant \frac{C}{(1-|z|^2)^{n+1+\alpha}}.$$

Hence

$$\left|\frac{\partial^N f}{\partial z^\beta}(z)\right|\leqslant \frac{1}{(1-|z|^2)^N}\Phi^{-1}\left(\frac{C}{(1-|z|^2)^{n+1+\alpha}}\right)\|f\|_{A_\alpha^\Phi(B^n)}^{lux} \qquad (4)$$

Now we consider the case where $\Phi\in\mathfrak{J}_p$. We recall that in this case Φ is of lower type $0<p\leqslant 1$. Let $\gamma>-1$ be large enough. As above, we have

$$\left|\frac{\partial^N f}{\partial z^\beta}(z)\right|\leqslant C\int_{B^n}\frac{|f(w)|}{|1-\langle z,w\rangle|^{n+1+N+\gamma}}dv_\alpha(w) \qquad (5)$$

We assume that $\gamma=\frac{n+1+\delta}{p}-(n+1)$ with $\delta>\alpha+p$. Then using Lemma 2.15 in Ref. [9], we obtain from (5) that

$$\left|\frac{\partial^N f}{\partial z^\beta}(z)\right|^p\leqslant C\int_{B^n}\left|\frac{f(w)}{(1-\langle z,w\rangle)^{n+1+N+\gamma}}\right|^p dv_\delta(w),$$

which leads to

$$\left|\frac{(1-|z|^2)^N}{\|f\|_{A_\alpha^\Phi(B^n)}^{lux}}\frac{\partial^N f}{\partial z^\beta}(z)\right|^p\leqslant C\int_{B^n}\left|\frac{f(w)}{\|f\|_{A_\alpha^\Phi(B^n)}^{lux}}\right|^p\cdot\frac{(1-|z|^2)^{Np}}{|1-\langle z,w\rangle|^{(n+1+N+\gamma)p}}dv_\delta(w) \qquad (6)$$

Also by Proposition 1.4.10 in Ref. [8], we see that $\frac{(1-|z|^2)^{Np}}{|1-\langle z,w\rangle|^{(n+1+N+\gamma)p}}dv_\delta(w)$ is up to a constant a probability measure. Hence using that the function $\Phi_p(t)=\Phi(t^{\frac{1}{p}})$ is convex and Jensen's inequality, we obtain that

$$\Phi_p\left(\left|\frac{(1-|z|^2)^N}{\|f\|_{A_\alpha^\Phi(B^n)}^{lux}}\frac{\partial^N f}{\partial z^\beta}(z)\right|^p\right)\leqslant C\int_{B^n}\Phi_p\left(\left|\frac{f(w)}{\|f\|_{A_\alpha^\Phi(B^n)}^{lux}}\right|^p\right)\cdot\frac{(1-|z|^2)^{Np}}{|1-\langle z,w\rangle|^{(n+1+N+\gamma)p}}dv_\delta(w).$$

From this and $\gamma=\frac{n+1+\delta}{p}-(n+1)$, it follows that

$$\Phi\left(\left|\frac{(1-|z|^2)^N}{\|f\|_{A_\alpha^\Phi(B^n)}^{lux}}\frac{\partial^N f}{\partial z^\beta}(z)\right|\right)\leqslant$$

$$C\int_{B^n}\Phi\left(\left|\frac{f(w)}{\|f\|^{lux}_{A^\Phi_\alpha(B^n)}}\right|\right)\cdot \frac{(1-|z|^2)^{Np}}{|1-\langle z,w\rangle|^{(n+1+N+\gamma)p}}dv_\delta(w)\leqslant$$

$$C\int_{B^n}\Phi\left(\left|\frac{f(w)}{\|f\|^{lux}_{A^\Phi_\alpha(B^n)}}\right|\right)\cdot \frac{(1-|z|^2)^{Np}}{|1-\langle z,w\rangle|^{n+1+\delta+Np}}dv_\alpha(w)\leqslant$$

$$\frac{C}{(1-|z|^2)^{n+1+\alpha}}\int_{B^n}\Phi\left(\left|\frac{f(w)}{\|f\|^{lux}_{A^\Phi_\alpha(B^n)}}\right|\right)dv_\alpha(w)$$

$$\leqslant\frac{C}{(1-|z|^2)^{n+1+\alpha}}.$$

Hence, in this case, we also have

$$\left|\frac{\partial^N f}{\partial z^\beta}(z)\right|\leqslant \frac{1}{(1-|z|^2)^N}\Phi^{-1}\left(\frac{C}{(1-|z|^2)^{n+1+\alpha}}\right)\|f\|^{lux}_{A^\Phi_\alpha(B^n)} \tag{7}$$

It is easy to see that the orders of all the possible partial derivatives of $f(z)$ in the expression of $\Re^m f(z)$ are not more than m. From this, (4) and (7), the desired result follows.

The following two results provide examples of useful functions of $A^\Phi_\alpha(B^n)$. The first was obtained in Ref. [2].

Lemma 2.3 Let $\alpha>-1$ and $\Phi\in\mathfrak{S}^q\cup\mathfrak{I}_p$. Then for every $t\geq 0$ and $w\in B^n$, the following function is in $A^\Phi_\alpha(B^n)$:

$$k_{w,t}(z)=\Phi^{-1}\left(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\right)\left(\frac{1-|w|^2}{1-\langle z,w\rangle}\right)^{2(n+1+\alpha)+t},$$

where C is an arbitrary positive constant. Moreover, $\sup\limits_{w\in B^n}\|k_{w,t}\|^{lux}_{A^\Phi_\alpha(B^n)}<1$.

By using the linear combinations of $k_{w,t}$, we obtain the following result.

Lemma 2.4 Let $w\in B^n$. Then for each fixed $k\in\{1,2,\cdots,l\}$, there exist constants $c_{k,1},c_{k,2},\cdots,c_{k,l}$ such that the function

$$f_{w,k}(z)=\sum_{i=1}^{l}c_{k,i}k_{w,i}(z)$$

satisfies

$$\Re^k f_{w,k}(w)=\Phi^{-1}\left(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\right)\frac{|w|^{2k}}{(1-|w|^2)^k}$$

and

$$\Re^j f_{w,k}(w)=0 \tag{8}$$

for each $j\in\{1,2,\cdots,l\}\setminus\{k\}$. Moreover, $\sup\limits_{w\in B^n}\|f_{w,t}\|^{lux}_{A^\Phi_\alpha(B^n)}<1$.

Proof Let $\beta=2(n+1+\alpha)$ and $\Phi_\alpha=\Phi^{-1}\left(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\right)$. From a direct calculation, we have

$$\Re k_{w,i}(w)=\Phi_\alpha(\beta+i)\frac{|w|^2}{1-|w|^2} \tag{9}$$

$$\Re^2 k_{w,i}(w)=\Phi_\alpha\left[(\beta+i)(\beta+i+1)\frac{|w|^4}{(1-|w|^2)^2}+(\beta+i)\frac{|w|^2}{1-|w|^2}\right] \tag{10}$$

and

$$\Re^s k_{w,i}(w)=\Phi_\alpha\left[\prod_{j=0}^{s-1}(\beta+i+j)\frac{|w|^{2s}}{(1-|w|^2)^s}+a_{s,s-1}\prod_{j=0}^{s-2}(\beta+i+j)\frac{|w|^{2(s-1)}}{(1-|w|^2)^{s-1}}+\cdots+a_{s,2}(\beta+i)(\beta+i+1)\frac{|w|^4}{(1-|w|^2)^2}+(\beta+i)\frac{|w|^2}{1-|w|^2}\right] \tag{11}$$

for each $s\geq 3$, where $a_{s,2},\cdots,a_{s,s-1}$ are some positive integers. From (9)～(11), it follows that (8) is equivalent to the following system

$$\begin{cases}\sum_{i=1}^{l}(\beta+i)c_{k,i}=0,\\ \sum_{i=1}^{l}(\beta+i)(\beta+i+1)c_{k,i}=0,\\ \cdots\cdots\\ \sum_{i=1}^{l}\prod_{j=0}^{k-1}(\beta+i+j)c_{k,i}=1,\\ \cdots\cdots\\ \sum_{i=1}^{l}\prod_{j=0}^{l-1}(\beta+i+j)c_{k,i}=0\end{cases} \tag{12}$$

Hence we only need to prove that there exist constants $c_{k,1},c_{k,2},\cdots,c_{k,n+1}$ such that the system (12) holds. By Lemma 3 in Ref. [12], the determinant of the system (12) equals to $\prod_{j=1}^{l}j!$ which is different from zero. Therefore, there exist constants $c_{k,1},c_{k,2},\cdots,c_{k,l}$ such that the system (12) holds. From Lemma 2.5, the asymptotic estimate follows.

3 Boundedness and compactness of $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$

We first characterize the boundedness of $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$.

Theorem 3.1 Let $\alpha>-1, u\in H(B^n), \Phi\in \aleph^q\cup\mathfrak{I}_p$, and ω a positive function defined on $(0,1]$. Then $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$ is bounded if and only if

$$M:=\sup_{z\in B^n}\frac{|u(z)|}{\omega(1-|z|)(1-|z|^2)^m}\cdot \Phi^{-1}\left(\frac{D}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right)<\infty \tag{13}$$

where D is the positive constant in Lemma 2.2.

Proof Suppose that $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$ is bounded. Since Lemma 2.4 holds for an arbitrary positive constant, here we take D the positive constant in Lemma 2.2. Then by Lemma 2.4, we have

$$\frac{|\mathfrak{R}_u^m f_{w,m}(w)|}{\omega(1-|w|)}=\Phi^{-1}\left(\frac{D}{(1-|w|^2)^{n+1+\alpha}}\right)\frac{|u(z)|\,|w|^{2m}}{\omega(1-|z|)(1-|z|^2)^m}\leqslant \|\mathfrak{R}_u^m f_{w,m}(w)\|_{H_w^\infty(B^n)}\leqslant C\ \|\mathfrak{R}_u^m\|_{H_w^\infty(B^n)},$$

from which it follows that

$$\sup_{|z|>\frac{1}{2}}\Phi^{-1}\left(\frac{D}{(1-|z|^2)^{n+1+\alpha}}\right)\cdot \frac{|u(z)|}{\omega(1-|z|)(1-|z|^2)^m}\leqslant C\ \|\mathfrak{R}_u^m\|_{H_w^\infty(B^n)}<+\infty \tag{14}$$

On the other hand, it is easy to see that

$$\sup_{|z|\leqslant\frac{1}{2}}\Phi^{-1}\left(\frac{D}{(1-|z|^2)^{n+1+\alpha}}\right)\cdot \frac{|u(z)|}{\omega(1-|z|)(1-|z|^2)^m}\leqslant C\max_{|z|=\frac{1}{2}}|u(z)|<+\infty \tag{15}$$

where

$$C=\Phi^{-1}\left(\left(\frac{4}{3}\right)^{n+1+\alpha}\right)\left(\frac{4}{3}\right)^m\frac{1}{\omega(1)}.$$

Hence from (14) and (15), (13) follows.

Suppose that (13) holds, that is,

$$M=\sup_{z\in B^n}\frac{|u(z)|}{\omega(1-|z|)(1-|z|^2)^m}\cdot \Phi^{-1}\left(\frac{D}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right)<\infty.$$

Then for every $f\in A_\alpha^\Phi(B^n)$, from Lemma 2.2 we have

$$\|\mathfrak{R}_u^m f\|_{H_w^\infty(B^n)}=\sup_{z\in B^n}\frac{|u(z)|}{\omega(1-|z|)}|\mathfrak{R}^m f(z)|\leqslant CM\ \|f\|_{A_\alpha^\Phi(B^n)}^{lux},$$

which shows that $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$ is bounded.

From the fact $H_\omega^\infty(B^n)\to\Lambda_\omega(B^n)$ when $\omega\in\Omega_2$, and Theorem 3.1, we can obtain the following result.

Proposition 3.2 Let $\alpha>-1, u\in H(B^n), \Phi\in\aleph^q\cup\mathfrak{I}_p$, and $\omega\in\Omega_2$. If $M<\infty$, then $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$ is bounded.

Theorem 3.3 Let $\alpha>-1, u\in H(B^n), \Phi\in\aleph^q\cup\mathfrak{I}_p$, and ω a positive function defined on $(0,1]$. Then $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$ is compact if and only if

$$\lim_{|z|\to 1}\frac{|u(z)|}{\omega(1-|z|)(1-|z|^2)^m}\cdot \Phi^{-1}\left(\frac{D}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right)=0 \tag{16}$$

where D is the positive constant in Lemma 2.2.

Proof Suppose that $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$ is compact. Consider a sequence $\{z_j\}$ in B^n such that $|z_j|\to 1$ as $j\to\infty$. If such sequence does not exist, then (16) obviously holds. Using this sequence, we define the functions $f_j(z)=f_{z_j,m}(z)$. Then the sequence $\{f_j\}$ is uniformly bounded in $A_\alpha^\Phi(B^n)$ and uniformly converges to zero on any compact subset of B^n as $j\to\infty$. Similar to the proof of Theorem 3.1, we have

$$\frac{|u(z_j)|\,|z_j|^{2m}}{\omega(1-|z_j|)(1-|z_j|^2)^m}\cdot \Phi^{-1}\left(\frac{D}{(1-|z_j|^2)^{n+1+\alpha}}\right)\leqslant \|\mathfrak{R}_u^m f_j\|_{H_w^\infty(B^n)} \tag{17}$$

From (17) and Lemma 2.1, (16) holds.

Now suppose that (16) holds. We first check that $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$ is bounded. For this, we observe that (17) implies that for every $\varepsilon>0$, there is an $\eta\in(0,1)$ such that

$$\frac{|u(z)|}{\omega(1-|z|)(1-|z|^2)^m}\cdot \Phi^{-1}\left(\frac{D}{(1-|z|^2)^{n+1+\alpha}}\right)<\varepsilon \tag{18}$$

for any $z\in K_\eta=\{z\in B^n:|z|>\eta\}$. Write

$$I(z):=\frac{|u(z)|}{\omega(1-|z|)(1-|z|^2)^m}\cdot \Phi^{-1}\left(\frac{D}{(1-|z|^2)^{n+1+\alpha}}\right).$$

Then from (18) we have

$$M=\sup_{z\in B^n} I(z)=\sup_{z\in B^n\setminus K_\eta} I(z)+\sup_{z\in K_\eta} I(z)\leqslant \frac{1}{\omega(1-\eta)(1-\eta^2)^m}\Phi^{-1}\cdot \left(\frac{D}{(1-\eta^2)^{n+1+\alpha}}\right)\max_{|z|=\eta}|u(z)|+\varepsilon.$$

From this and Theorem 3.1, it follows that $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$ is bounded.

To prove that $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$ is compact, by Lemma 2.1 we just need to prove that if $\{f_j\}$ is a sequence in $A_\alpha^\Phi(B^n)$ such that $\|f_j\|_{A_\alpha^\Phi(B^n)}\leqslant M$ and $\{f_j\}$ uniformly converges to zero on any compact subset of B^n as $j\to\infty$, then

$$\lim_{j\to\infty}\|\mathfrak{R}_u^m f_j\|_{H_\omega^\infty(B^n)}=0.$$

For any $\varepsilon>0$ and the associated η in (18), by using Lemma 2.2, we have

$$\begin{aligned}\|\mathfrak{R}_u^m f_j\|_{H_\omega^\infty(B^n)}=\ &\sup_{z\in B^n}\frac{1}{\omega(1-|z|)}|u(z)\mathfrak{R}^m f_j(z)|=\\ &\sup_{z\in B^n\setminus K_\eta}\frac{|u(z)|}{\omega(1-|z|)}|\mathfrak{R}^m f_j(z)|+\\ &\sup_{z\in K_\eta}\frac{|u(z)|}{\omega(1-|z|)}|\mathfrak{R}^m f_j(z)|\leqslant\\ &\frac{1}{\omega(1-\eta)}\max_{\{z:|z|\leqslant\eta\}}|u(z)|\sup_{\{z:|z|\leqslant\eta\}}|\mathfrak{R}^m f_j(z)|+\\ &C\sup_{\{z:|z|>\eta\}}\frac{|u(z)|}{\omega(1-|z|)(1-|z|^2)^m}\cdot\\ &\Phi^{-1}\left(\frac{D}{(1-|z|^2)^{n+1+\alpha}}\right)\leqslant\\ &\frac{1}{\omega(1-\eta)}\max_{\{z:|z|\leqslant\eta\}}|u(z)|\sup_{\{z:|z|\leqslant\eta\}}|\mathfrak{R}^m f_j(z)|+\\ &C\varepsilon \qquad (19)\end{aligned}$$

It is easy to see that, if $\{f_j\}$ uniformly converges to zero on any compact subset of B^n, then $\left\{\frac{\partial^N f_j}{\partial z^\beta}\right\}$ also does as $j\to\infty$. This shows that $\{|\mathfrak{R}^m f_j|\}$ uniformly converges to zero on any compact subset of B^n as $j\to\infty$. Since $\{z\in B^n:|z|\leqslant\eta\}$ is compact subset of B^n, by letting $j\to\infty$ in (19) we have

$$\lim_{j\to\infty}\|\mathfrak{R}_u^m f_j\|_{H_\omega^\infty(B^n)}=0.$$

This shows that $\mathfrak{R}_u^m: A_\alpha^\Phi(B^n)\to H_\omega^\infty(B^n)$ is compact.

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