doi：10.3969/j.issn.0490-6756.2020.01.003

# $S_1$-Frankl 猜想的 6 元素情形(Ⅰ)

胡泽春，李世伦
(四川大学数学学院，成都 610064)

**摘　要**：并封闭集猜想(又称 Frankl 猜想)即对于一个由有限个有限集合构成的关于并运算封闭的集族，如果这个集族至少包含一个非空集合，那么存在一个元素包含在至少一半的集合里. 最近，本文作者与崔振提出了 Frankl 猜想的两个加强版本(简称为 $S_1$-Frankl 猜想与 $S_2$-Frankl 猜想)，并给出了部分证明. 特别地，作者证明：如果 $n\leqslant 5$，则 $S_1$-Frankl 猜想成立，其中 $n$ 表示这个集族中所有元素的个数. 本文及其姊妹文证明当 $n=6$ 时结论也成立. 这是证明的第一部分.

**关键词**：Frankl 猜想；并封闭集猜想
**中图分类号**：Q157.1；O144.1　　**文献标识码**：A　　**文章编号**：0490-6756(2020)01-0011-16

# The 6-element case of $S_1$-Frankl conjecture (Ⅰ)

*HU Ze-Chun*, *LI Shi-Lun*
(School of Mathematics, Sichuan University, Chengdu 610064, China)

**Abstract**: The union-closed sets conjecture (Frankl's conjecture) says that for any finite union-closed family of finite sets, other than the family consisting only of the empty set, there exists an element that belongs to at least half of the sets in the family. Recently, two stronger versions of Frankl's conjecture ($S_1$-Frankl conjecture and $S_2$-Frankl conjecture for short) were introduced and partial proofs were given. In particular, it was proved that $S_1$-Frankl conjecture holds if $n\leqslant 5$, where $n$ is the number of all the elements in the family of sets. In this paper and its sister paper, we prove that it holds if $n=6$. This is the first part of the proof.

**Keywords**: Frankl's conjecture; Union-closed sets conjecture
(2010 MSC 05A05; 05A99)

## 1 Introduction

A family $F$ of sets is union-closed if $A,B\in F$ implies $A\cup B\in F$. For simplicity, denote $n=|\cup_{A\in F}A|$ and $m=|F|$.

In 1979, Frankl[1-2] conjectured that for any finite union-closed family of finite sets, other than the family consisting only of the empty set, there exists an element that belongs to at least half of the sets in the family. If a union-closed family $F$ contains a set with one element or two elements, then Frankl's conjecture holds for $F$[3]. The result was extended by Poonen[4]. In addition, Ref. [4] proved that Frankl's conjecture holds if $n\leqslant 7$ or $m\leqslant 28$, and presented an equivalent lattice formulation of Frankl's conjecture.

**收稿日期**：2018-09-06
**基金项目**：国家自然科学基金(11771309)；中央高校基本科研业务费
**作者简介**：胡泽春(1976－)，男，湖北利川人，教授，主要研究方向为随机过程，极限理论，组合数学等. E-mail：zchu@scu.edu.cn
**通讯作者**：李世伦. E-mail：lishilun@scu.edu.cn

Bošnjak and Marković[5] proved that Frankl's conjecture holds if $n \leqslant 11$. Zivković and Vučković gave a computer assisted proof in "The 12-element case of Frankl's conjecture, Preprint (2012)" that Frankl's conjecture holds if $n \leqslant 12$, which together with Faro's result[6] (see also Roberts and Simposon[7]) implies that Frankl's conjecture is true if $m \leqslant 50$. For more progress on Frankl's conjecture, we refer to Refs. [3,8-16].

Let $M_n=\{1,2,\cdots,n\}$ and $F \subset 2^{M_n}=\{A: A \subset M_n\}$ with $\cup_{A \in F} A=M_n$. Suppose that $F$ is union-closed. Without loss of generality, we assume that $\varnothing \in F$. For any $k=1,2,\cdots,n$, define

$$\boldsymbol{M}_k=\{A \in 2^{M_n}:|A|=k\},$$

and

$$T(F)=\inf\{1 \leqslant k \leqslant n: F \cap \boldsymbol{M}_k \neq \varnothing\}.$$

Then $1 \leqslant T(F) \leqslant n$. By virtue of $T(F)$, Ref. [17] introduced the following two stronger versions of Frankl's conjecture:

$S_1$-Frankl conjecture: If $n \geqslant 2$ and $T(F)=k \in\{2,\cdots,n\}$, then there exist at least $k$ elements in $M_n$ which belong to at least half of the sets in $F$;

$S_2$-Frankl conjecture: If $n \geqslant 2$ and $T(F)=k \in\{2,\cdots,n\}$, then there exist at least two elements in $M_n$ which belong to at least half of the sets in $F$.

We need the following lemma, which has been used in some proofs in Ref. [17].

**Lemma 1.1** Suppose that $M$ is a finite set with $|M| \geqslant 2$ and $G \subset\{A \subset M:|A|=|M|-1\}$. If $|G| \geqslant 2$, then all the elements in $M$ belong to at least $|G|-1$ set(s) in $G$.

**Proof** Without loss of generality, we assume that $M=M_n=\{1,2,\cdots,n\}$ with $n \geqslant 2$. Then $G$ is a subset of $\boldsymbol{M}_{n-1}=\{A \subset M_n:|A|=n-1\}$. Notice that for any $i \in\{1,2,\cdots,n\}$, it belongs to all the sets in $\boldsymbol{M}_{n-1}$ except the set $M_n \backslash\{i\}$. Hence all the elements in $M$ belong to at least $|G|-1$ set(s) in $G$.

When we consider $S_1$-Frankl conjecture for the case that $n=6$, by section 2 of Ref. [17], we know that if $T(F) \in\{4,5,6\}$, then there exist at least $T(F)$ elements in $M_6$ which belong to at least half of the sets in $F$. Thus we need only to consider the two cases $T(F)=3$ and $T(F)=2$. In next section, we will prove that $S_1$-Frankl conjecture holds when $n=6$ and $T(F)=3$. The proof for the case that $n=6$ and $T(F)=2$ will be given in a sister paper.

## 2 $S_1$-Frankl conjecture for $n=6$ and $T(F)=3$

Let $M_6=\{1,2,\cdots,6\}$ and $F \subset 2^{M_6}=\{A: A \subset M_6\}$ with $\cup_{A \in F} A=M_6$. Suppose that $F$ is union-closed and $\varnothing \in F$. For $k=1,2,\cdots,6$, define

$$\boldsymbol{M}_k=\{A \in 2^{M_6}:|A|=k\}, n_k=|F \cap \boldsymbol{M}_k|,$$

and

$$T(F)=\inf\{1 \leqslant k \leqslant 6: n_k>0\}.$$

Then $1 \leqslant T(F) \leqslant 6$.

In the following, we assume that $T(F)=3$, and will prove that there exist at least 3 elements in $M_6$ which belong to at least half of the sets in $F$. We have 4 cases: $F=\{\varnothing, M_6\} \cup \boldsymbol{G}_3$, $F=\{\varnothing, M_6\} \cup \boldsymbol{G}_3 \cup \boldsymbol{G}_5$, $F=\{\varnothing, M_6\} \cup \boldsymbol{G}_3 \cup \boldsymbol{G}_4$ and $F=\{\varnothing, M_6\} \cup \boldsymbol{G}_3 \cup \boldsymbol{G}_4 \cup \boldsymbol{G}_5$, where $\boldsymbol{G}_i$ is a nonempty subset of $M_i$ for $i=3,4,5$.

### 2.1 $F=\{\varnothing, \boldsymbol{M}_6\} \cup \boldsymbol{G}_3$

We have two subcases: $n_3=1$ and $n_3 \geqslant 2$. Throughout the rest of this paper, we omit the sentences of this type. Denote $\mathbf{G}_3=\{G_1, \cdots, G_{n_3}\}$.

(1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$. Then all the 3 elements in $G_1$ belong to two sets among the three sets in $F$.

(2) $n_3 \geqslant 2$. For any $i, j=1, \cdots, n_3, i \neq j$, we must have $G_i \cup G_j=M_6$, which implies that $G_i \cap G_j=\varnothing$. Hence $n_3=2$. Now all the 6 elements in $M_6$ belong to two sets among the four sets in $F$.

### 2.2 $F=\{\varnothing, \boldsymbol{M}_6\} \cup \boldsymbol{G}_3 \cup \boldsymbol{G}_5$

Denote $\boldsymbol{G}_3=\{G_1, \cdots, G_{n_3}\}$ and $\boldsymbol{G}_5=\{H_1, \cdots, H_{n_5}\}$.

(1) $n_5=1$. Now $G_5=\{H_1\}$. Without loss of generality, we assume that $H_1=\{1,2,3,4,5\}$.

(1.1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$. Notice that all the elements in $G_1 \cup\{1,2,3,4,5\}$ belong to at least one of the two sets $G_1$ and $\{1,2,3,4,5\}$. Then we know that all the elements in $G_1 \cup\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(1.2) $n_3 \geqslant 2$. For any $i,j=1,\cdots,n_3, i\neq j$, we have $G_i\cup G_j=M_6$ or $G_i\cup G_j=\{1,2,3,4,5\}$.

(1.2.1) $n_3$ is an even number and there is a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-1}}\cup G_{i_{n_3}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\boldsymbol{G}_3$ and thus all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(1.2.2) $n_3$ is an odd number and there is a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-2}}\cup G_{i_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{n_3-1}}\}$ and thus all the elements in $G_{i_{n_3}}\cup\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(1.2.3) We can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{i_1},\cdots,G_{i_{2k}}\}$ (hereafter this part may be an empty set) and $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$, where $\{i_1,\cdots,i_{n_3}\}=\{1,\cdots,n_3\}$, $n_3-2k\geqslant 2$, and

(i) $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{2k-1}}\cup G_{i_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{i_{2k+1},\cdots,i_{n_3}\}$, $G_i\cup G_j=\{1,2,3,4,5\}$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{2k}}\}$. Without loss of generality, we assume that $G_{i_{2k+1}}=\{1,2,3\}$. By (ii), we know that for any $j=i_{2k+2},\cdots,i_{n_3}$,

$$G_j\in\{\{1,4,5\},\{2,4,5\},\{3,4,5\}\}.$$

Since $|\{1,4,5\}\cup\{2,4,5\}|=|\{1,4,5\}\cup\{3,4,5\}|=|\{2,4,5\}\cup\{3,4,5\}|=4$, by (ii) again, we know that in this case $n_3-2k=2$. Then we know that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(2) $n_5\geqslant 2$. By Lemma 1.1, we know that all the 6 elements in $M_6$ belong to at least $n_5-1$ set(s) in $\boldsymbol{G}_5$ and thus belong to at least half of the sets in $\boldsymbol{G}_5$. Hence it is enough to show that there exist at least 3 elements in $M_6$ which belong to at least half of the sets in $\boldsymbol{G}_3$.

(2.1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$. Then all the 3 elements in $G_1$ satisfy the condition.

(2.2) $n_3\geqslant 2$. For any $i,j=1,\cdots,n_3, i\neq j$, we have $G_i\cup G_j=M_6$ or $|G_i\cup G_j|=5$.

(2.2.1) $n_3$ is an even number and there is a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-1}}\cup G_{i_{n_3}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\boldsymbol{G}_3$.

(2.2.2) $n_3$ is an odd number and there is a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-2}}\cup G_{i_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{n_3-1}}\}$ and thus all the 3 elements in $G_{i_{n_3}}$ belong to at least half of the sets in $\boldsymbol{G}_3$.

(2.2.3) We can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{i_1},\cdots,G_{i_{2k}}\}$ and $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$, where $\{i_1,\cdots,i_{n_3}\}=\{1,\cdots,n_3\}$, $n_3-2k\geqslant 2$, and

(i) $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{2k-1}}\cup G_{i_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{i_{2k+1},\cdots,i_{n_3}\}$, $|G_i\cup G_j|=5$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{2k}}\}$. Without loss of generality, we assume that $G_{i_{2k+1}}=\{1,2,3\}$. By (ii), we know that for any $j=i_{2k+2},\cdots,i_{n_3}$,

$$G_j\in\{\{1,4,5\},\{2,4,5\},\{3,4,5\},\{1,4,6\},\{2,4,6\},\{3,4,6\},\{1,5,6\},\{2,5,6\},\{3,5,6\}\}.$$

Since $|\{1,4,5\}\cup\{2,4,5\}|=|\{1,4,5\}\cup\{3,4,5\}|=|\{2,4,5\}\cup\{3,4,5\}|=4$, by (ii) again, we know that $|\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}\cap\{\{1,4,5\},\{2,4,5\},\{3,4,5\}\}|\leqslant 1$. Similarly, we have that

$$|\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}\cap\{\{1,4,6\},\{2,4,6\},\{3,4,6\}\}|\leqslant 1,$$

$$|\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}\cap\{\{1,5,6\},\{2,5,6\},\{3,5,6\}\}|\leqslant 1.$$

Hence we need only to consider the following 3 cases.

(2.2.3.1) $n_3-2k=2$. Take $|\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}\cap\{\{1,4,5\},\{2,4,5\},\{3,4,5\}\}|=1$ for example. Without loss of generality, we assume that $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}=\{\{1,2,3\},\{1,4,5\}\}$. Now all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $\boldsymbol{G}_3$.

(2.2.3.2) $n_3-2k=3$. Take $|\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}\cap\{\{1,4,5\},\{2,4,5\},\{3,4,5\}\}|=1=|\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}\cap\{\{1,4,6\},\{2,4,6\},\{3,4,6\}\}|$ for example. Without loss of generality, we assume that $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}=\{\{1,2,3\},\{1,4,5\},\{2,4,6\}\}$. Now all the 3 elements in $\{1,2,4\}$ belong to at least half of the sets in $\boldsymbol{G}_3$.

(2.2.3.3) $n_3-2k=4$. Without loss of generality, we assume that $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}=\{\{1,2,3\},\{1,4,5\},\{2,4,6\},\{3,5,6\}\}$. Now all the 6 elements in $M_6$ belong to at least half of the sets in $\boldsymbol{G}_3$.

### 2.3 $F=\{\emptyset,M_6\}\cup G_3\cup G_4$

Denote $\boldsymbol{G}_3=\{G_1,\cdots,G_{n_3}\}$ and $\boldsymbol{G}_4=\{H_1,\cdots,H_{n_4}\}$.

(1) $n_4=1$. Now $\boldsymbol{G}_4=\{H_1\}$. Without loss of generality, we assume that $H_1=\{1,2,3,4\}$.

(1.1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$. Then all the elements in $G_1\cup\{1,2,3,4\}$ belong to at least half of the sets in $F$.

(1.2) $n_3\geqslant 2$. For any $i,j=1,\cdots,n_3,i\neq j$, we have $G_i\cup G_j=M_6$ or $G_i\cup G_j=\{1,2,3,4\}$.

(1.2.1) $n_3$ is an even number and there is a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-1}}\cup G_{i_{n_3}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\boldsymbol{G}_3$ and thus all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $F$.

(1.2.2) $n_3$ is an odd number and there is a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-2}}\cup G_{i_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{n_3-1}}\}$ and thus all the elements in $G_{i_{n_3}}\cup\{1,2,3,4\}$ belong to at least half of the sets in $F$.

(1.2.3) We can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{i_1},\cdots,G_{i_{2k}}\}$ and $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$, where $\{i_1,\cdots,i_{n_3}\}=\{1,\cdots,n_3\}$, $n_3-2k\geqslant 2$, and

(i) $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{2k-1}}\cup G_{i_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{i_{2k+1},\cdots,i_{n_3}\}$, $G_i\cup G_j=\{1,2,3,4\}$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{2k}}\}$. By Lemma 1.1, we know that all the 4 elements in $\{1,2,3,4\}$ belong to at least $n_3-2k-1$ set(s) in $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$ and thus belong to at least half of the sets in $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$. Hence in this case, all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $F$.

(2) $n_4\geqslant 2$. For any $i,j=1,\cdots,n_4,i\neq j$, we must have $H_i\cup H_j=M_6$.

We claim that all the 6 elements in $M_6$ belong to at least half of the sets in $\boldsymbol{G}_4$. In fact, if $n_4=2k$ is an even number, then $H_1\cup H_2=\cdots=H_{2k-1}\cup H_{2k}=M_6$ and thus all the 6 elements in $M_6$ belong to at least half of the sets in $\boldsymbol{G}_4$. If $n_4=2k+1$ is an odd number, then by

$$H_1\cup H_2=\cdots=H_{2k-1}\cup H_{2k}=M_6,$$

we know that all the 4 elements in $H_{2k+1}$ belong to at least half of the sets in $G_4$; by

$$H_2\cup H_3=\cdots=H_{2k}\cup H_{2k+1}=M_6,$$

we know that all the 4 elements in $G_1$ belong to at least half of the sets in $\boldsymbol{G}_4$; by $H_1\cup H_{2k+1}=M_6$, we know that all the 6 elements in $M_6$ belong to at least half of the sets in $\boldsymbol{G}_4$. Hence it is enough to show that there exist 3 elements in $M_6$ which belong to at least half of the sets in $\boldsymbol{G}_3$.

(2.1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$. Then all the 3 elements in $G_1$ satisfy the condition.

(2.2) $n_3\geqslant 2$. For any $i,j=1,\cdots,n_3,i\neq j$, we have $G_i\cup G_j=M_6$ or $|G_i\cup G_j|=4$.

(2.2.1) $n_3$ is an even number and there is a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-1}}\cup G_{i_{n_3}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\boldsymbol{G}_3$.

(2.2.2) $n_3$ is an odd number and there is a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-2}}\cup G_{i_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{n_3-1}}\}$ and thus all the 3 elements in $G_{i_{n_3}}$ belong to at least half of the sets in $G_3$.

(2.2.3) We can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{i_1},\cdots,G_{i_{2k}}\}$ and $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$, where $\{i_1,\cdots,i_{n_3}\}=\{1,\cdots,n_3\}$, $n_3-2k\geqslant 2$, and

(i) $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{2k-1}}\cup G_{i_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{i_{2k+1},\cdots,i_{n_3}\}$, $|G_i\cup G_j|=4$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{2k}}\}$. Without loss of generality, we assume that $G_{i_{2k+1}}=\{1,2,3\}$. By (ii), we know that for any $j=i_{2k+2},\cdots,i_{n_3}$, $G_j\in\{\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,2,6\},\{1,3,6\},\{2,3,6\}\}$.

Denote

$$H_4=\{\{1,2,4\},\{1,3,4\},\{2,3,4\}\},$$

$$H_5=\{\{1,2,5\},\{1,3,5\},\{2,3,5\}\},$$

$$H_6=\{\{1,2,6\},\{1,3,6\},\{2,3,6\}\}.$$

For any $A\in H_4, B\in H_5, C\in H_6$, we have

$\{1,2,3\}\cup A\cup B=\{1,2,3,4,5\}$,

$\{1,2,3\}\cup A\cup C=\{1,2,3,4,6\}$,

$\{1,2,3\}\cup B\cup C=\{1,2,3,5,6\}$.

By $F=\{\varnothing, M_6\}\cup \boldsymbol{G}_3\cup \boldsymbol{G}_4$, without loss of generality, we can assume that

$$F\cap H_4\neq\varnothing, F\cap H_5=F\cap H_6=\varnothing.$$

Now, by Lemma 1.1, we know that all the 4 elements in $\{1,2,3,4\}$ belong to at least $n_3-2k-1$ set(s) in $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$, and thus belong to at least half of the sets in $G_3$.

**2.4　$F=\{\varnothing, M_6\}\cup G_3\cup G_4\cup G_5$**

Denote $\boldsymbol{G}_3=\{G_1,\cdots,G_{n_3}\}, \boldsymbol{G}_4=\{H_1,\cdots,H_{n_4}\}$ and $\boldsymbol{G}_5=\{I_1,\cdots,I_{n_5}\}$.

(1) $n_5=1$. Now $\boldsymbol{G}_5=\{I_1\}$. Without loss of generality, we assume that $I_1=\{1,2,3,4,5\}$.

(1.1) $n_4=1$. Now $\boldsymbol{G}_4=\{H_1\}$.

(1.1.1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$. If $H_1\subset I_1$, then all the 4 elements in $H_1$ belong to at least two sets among the three sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4\cup\boldsymbol{G}_5$ and thus belong to at least half of the sets in $F$. If $H_1\not\subset I_1$, then $H_1\cup I_1=M_6$ and thus in this case all the 3 elements in $G_1$ belong to at least half of the sets in $F$.

(1.1.2) $n_3\geqslant 2$. Now for any $i,j=1,\cdots,n_3, i\neq j$, we have $G_i\cup G_j=M_6$ or $G_i\cup G_j=I_1=\{1,2,3,4,5\}$ or $G_i\cup G_j=H_1$.

(1.1.2.1) $n_3$ is an even number and there exists a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-1}}\cup G_{i_{n_3}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\boldsymbol{G}_3$. Hence all the elements in $H_1\cup I_1$ belong to at least half of the sets in $F$.

(1.1.2.2) $n_3$ is an odd number and there exists a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-2}}\cup G_{i_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{n_3-1}}\}$. If $H_1\subset I_1$, then all the 4 elements in $H_1$ belong to at least two sets among the three sets in $\{G_{i_{n_3}}, H_1, I_1\}$ and thus belong to at least half of the sets in $F$. If $H_1\not\subset I_1$, then $H_1\cup I_1=M_6$ and thus in this case all the 3 elements in $G_{i_{n_3}}$ belong to at least half of the sets in $F$.

(1.1.2.3) We can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{i_1},\cdots,G_{i_{2k}}\}$ and $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$, where $\{i_1,\cdots,i_{n_3}\}=\{1,\cdots,n_3\}$, $n_3-2k\geqslant 2$, and

(i) $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{2k-1}}\cup G_{i_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{i_{2k+1},\cdots,i_{n_3}\}$, $G_i\cup G_j=I_1=\{1,2,3,4,5\}$ or $G_i\cup G_j=H_1$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{2k}}\}$.

(a) $H_1\subset\{1,2,3,4,5\}$. Without loss of generality, we assume that $H_1=\{1,2,3,4\}$.

(a.1) $n_3-2k$ is an even number and there exists a permutation $(j_1,\cdots,j_{n_3-2k})$ of $(i_{2k+1},\cdots,i_{n_3})$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-2k-1}}\cup G_{j_{n_3-2k}}=\{1,2,3,4,5\}$. Then all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$. Note that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least one of the two sets $H_1$ and $I_1$. Then we know that in this case, all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(a.2) $n_3-2k$ is an odd number and there exists a permutation $(j_1,\cdots,j_{n_3-2k})$ of $(i_{2k+1},\cdots,i_{n_3})$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-2k-2}}\cup G_{j_{n_3-2k-1}}=\{1,2,3,4,5\}$. Then all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $\{G_{j_1},\cdots,G_{j_{n_3-2k-1}}\}$. Note that all the 4 elements in $\{1,2,3,4\}$ belong to at least two sets among the three sets $\{G_{j_{n_3-2k}}, H_1, I_1\}$. Then we know that in this case, all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $F$.

(a.3) We can decompose $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$ into two disjoint parts $\{G_{j_1},\cdots,G_{j_{2l}}\}$ and $\{G_{j_{2l+1}},\cdots,G_{j_{n_3-2k}}\}$, where $\{j_1,\cdots,j_{n_3-2k}\}=\{i_{2k+1},\cdots,i_{n_3}\}$, $n_3-2k-2l\geqslant 2$, and

(iii) $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{2l-1}}\cup G_{j_{2l}}=\{1,2,3,4,5\}$;

(iv) for any two different indexes $\{i,j\}$ from $\{j_{2l+1},\cdots,j_{n_3-2k}\}$, $G_i\cup G_j=H_1=\{1,2,3,4\}$.

Then all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $\{G_{j_1},\cdots,G_{j_{2l}}\}$. By Lemma 1.1, we know that all the 4 elements in $\{1,2,3,4\}$ belong to at least $n_3-2k-2l-1$ set(s) in

$\{G_{j_{2l+1}},\cdots,G_{j_{n_3-2k}}\}$ and thus belong to at least half of the sets in $\{G_{j_{2l+1}},\cdots,G_{j_{n_3-2k}}\}$. Hence in this case, all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $F$.

(b) $H_1 \not\subseteq \{1,2,3,4,5\}$. Without loss of generality, we assume that $H_1=\{1,2,3,6\}$. By following the proof in (a), we can get that in this case all the 3 elements in $\{1,2,3\}$ belong to at least half of the sets in $F$.

(1.2) $n_4 \geqslant 2$. Now for any $i,j=1,\cdots,n_4, i\neq j$, we have $H_i \cup H_j = M_6$ or $H_i \cup H_j = I_1=\{1,2,3,4,5\}$.

(1.2.1) $n_4$ is an even number and there exists a permutation $(i_1,\cdots,i_{n_4})$ of $(1,\cdots,n_4)$ such that $H_{i_1}\cup H_{i_2}=\cdots=H_{i_{n_4-1}}\cup H_{i_{n_4}}=M_6$. Then all the 6 elements in $M_6$ belong to at least half of the sets in $\boldsymbol{G}_4$.

(1.2.1.1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$. In this case, all the elements in $G_1\cup I_1$ belong to at least half of the sets in $F$.

(1.2.1.2) $n_3\geqslant 2$. Now for any $i,j=1,\cdots,n_3, i\neq j$, we have $G_i\cup G_j=M_6$ or $G_i\cup G_j=I_1=\{1,2,3,4,5\}$ or $|G_i\cup G_j|=4$.

(a) $n_3$ is an even number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-1}}\cup G_{j_{n_3}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\boldsymbol{G}_3$. Hence in this case all the 5 elements in $I_1$ belong to at least half of the sets in $F$.

(b) $n_3$ is an odd number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-2}}\cup G_{j_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{n_3-1}}\}$. Hence in this case, all the elements in $G_{j_{n_3}}\cup I_1$ belong to at least half of the sets in $F$.

(c) We can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{j_1},\cdots,G_{j_{2k}}\}$ and $\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}$, where $\{j_1,\cdots,j_{n_3}\}=\{1,\cdots,n_3\}$, $n_3-2k\geqslant 2$, and

(i) $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{2k-1}}\cup G_{j_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{j_{2k+1},\cdots,j_{n_3}\}$, $G_i\cup G_j=I_1=\{1,2,3,4,5\}$ or $|G_i\cup G_j|=4$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{2k}}\}$.

(c.1) $n_3-2k$ is an even number and there exists a permutation $(m_1,\cdots,m_{n_3-2k})$ of $(j_{2k+1},\cdots,j_{n_3})$ such that $G_{m_1}\cup G_{m_2}=\cdots=G_{m_{n_3-2k-1}}\cup G_{m_{n_3-2k}}=\{1,2,3,4,5\}$. Hence in this case, all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(c.2) $n_3-2k$ is an odd number and there exists a permutation $(m_1,\cdots,m_{n_3-2k})$ of $(j_{2k+1},\cdots,j_{n_3})$ such that $G_{m_1}\cup G_{m_2}=\cdots=G_{m_{n_3-2k-2}}\cup G_{m_{n_3-2k-1}}=\{1,2,3,4,5\}$. Then all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $\{G_{m_1},\cdots,G_{m_{n_3-2k-1}}\}$. Note that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least one of the two sets $G_{m_{n_3-2k}}$ and $\{1,2,3,4,5\}$. Hence in this case, all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(c.3) We can decompose $\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}$ into two disjoint parts $\{G_{m_1},\cdots,G_{m_{2l}}\}$ and $\{G_{m_{2l+1}},\cdots,G_{m_{n_3-2k}}\}$, where $\{m_1,\cdots,m_{n_3-2k}\}=\{j_{2k+1},\cdots,j_{n_3}\}$, $n_3-2k-2l\geqslant 2$, and

(iii) $G_{m_1}\cup G_{m_2}=\cdots=G_{m_{2l-1}}\cup G_{m_{2l}}=\{1,2,3,4,5\}$;

(iv) for any two different indexes $\{i,j\}$ from $\{m_{2l+1},\cdots,m_{n_3-2k}\}$, $G_i\cup G_j\in G_4$.

Then all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $\{G_{m_1},\cdots,G_{m_{2l}}\}$.

(c.3.1) There exists $t\in\{m_{2l+1},\cdots,m_{n_3-2k}\}$ such that $G_t\subset\{1,2,3,4,5\}$. Without loss of generality, we assume that $G_{m_{2l+1}}=\{1,2,3\}$. Then for any $t\in\{m_{2l+2},\cdots,m_{n_3-2k}\}$, we have $G_t\in\boldsymbol{H}_4\cup\boldsymbol{H}_5\cup\boldsymbol{H}_6$, where

$\boldsymbol{H}_4=\{\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$,

$\boldsymbol{H}_5=\{\{1,2,5\},\{1,3,5\},\{2,3,5\}\}$,

$\boldsymbol{H}_6=\{\{1,2,6\},\{1,3,6\},\{2,3,6\}\}$.

For simplicity, define $\boldsymbol{H}:=\{G_{m_{2l+1}},\cdots,G_{m_{n_3-2k}}\}$. We have the following 7 cases.

(c.3.1.1) $\boldsymbol{H}\cap\boldsymbol{H}_4\neq\varnothing, \boldsymbol{H}\cap\boldsymbol{H}_5=\boldsymbol{H}\cap\boldsymbol{H}_6=\varnothing$.

(c.3.1.2) $\boldsymbol{H}\cap\boldsymbol{H}_5\neq\varnothing, \boldsymbol{H}\cap\boldsymbol{H}_4=\boldsymbol{H}\cap\boldsymbol{H}_6=\varnothing$.

(c.3.1.3) $\boldsymbol{H}\cap\boldsymbol{H}_6\neq\varnothing, \boldsymbol{H}\cap\boldsymbol{H}_4=\boldsymbol{H}\cap\boldsymbol{H}_5=\varnothing$.

(c.3.1.4) $\boldsymbol{H}\cap\boldsymbol{H}_4\neq\varnothing, \boldsymbol{H}\cap\boldsymbol{H}_5\neq\varnothing, \boldsymbol{H}\cap\boldsymbol{H}_6=\varnothing$.

(c.3.1.5) $\boldsymbol{H}\cap\boldsymbol{H}_4\neq\varnothing, \boldsymbol{H}\cap\boldsymbol{H}_6\neq\varnothing, \boldsymbol{H}\cap\boldsymbol{H}_5=\varnothing$.

(c. 3. 1. 6) $\boldsymbol{H}\cap\boldsymbol{H}_5\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_6\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_4=\varnothing$.

(c. 3. 1. 7) $\boldsymbol{H}\cap\boldsymbol{H}_4\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_5\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_6\neq\varnothing$.

As to (c. 3. 1. 1), by Lemma 1. 1, we know that all the 4 elements in $\{1,2,3,4\}$ belong to at least $n_3-2k-1$ set(s) in $\boldsymbol{H}$ and thus belong to at least half of the sets in $\boldsymbol{H}$. Hence in this case, all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $F$.

As to (c. 3. 1. 2) and (c. 3. 1. 3), we can get that all the 3 elements in $\{1,2,3\}$ belong to at least half of the sets in $F$.

As to (c. 3. 1. 4), without loss of generality, we assume that $\{1,2,4\}\in\boldsymbol{H}\cap\boldsymbol{H}_4$. Then by (iv), we know that $\boldsymbol{H}\cap\boldsymbol{H}_5=\{\{1,2,5\}\}$ and by (iv) again we get that $\boldsymbol{H}\cap\boldsymbol{H}_4=\{\{1,2,4\}\}$. Thus in this case $\boldsymbol{H}=\{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$. Note that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least two sets among the 4 sets in $\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,3,4,5\}\}$. Then we obtain that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

As to (c. 3. 1. 5), without loss of generality, we assume that $\{1,2,4\}\in\boldsymbol{H}\cap\boldsymbol{H}_4$. Then by (iv), we know that $\boldsymbol{H}\cap\boldsymbol{H}_6=\{\{1,2,6\}\}$ and by (iv) again we get that $\boldsymbol{H}\cap\boldsymbol{H}_4=\{\{1,2,4\}\}$. Thus in this case $\boldsymbol{H}=\{\{1,2,3\},\{1,2,4\},\{1,2,6\}\}$. By $\{1,2,3\}\cup\{1,2,4\}\cup\{1,2,6\}=\{1,2,3,4,6\}\in G_5=\{\{1,2,3,4,5\}\}$, we know that this case is impossible.

As to (c. 3. 1. 6) and (c. 3. 1. 7), by following the analysis to (c. 3. 1. 5), we know that these two cases are impossible.

(c. 3. 2) For any $t\in\{m_{2l+1},\cdots,m_{n_3-2k}\}$, $G_t\not\subseteq\{1,2,3,4,5\}$. Without loss of generality, we assume that $G_{m_{2l+1}}=\{1,2,6\}$. Then by (iv), we know that for any $t=m_{2l+2},\cdots,m_{n_3-2k}$, we have

$G_t\in\{\{1,3,6\},\{1,4,6\},\{1,5,6\},\{2,3,6\},\{2,4,6\},\{2,5,6\}\}$. Without loss of generality, we assume that $\{1,3,6\}\in\{G_{m_{2l+2}},\cdots,G_{m_{n_3-2k}}\}$. Then by $\boldsymbol{G}_5=\{\{1,2,3,4,5\}\}$, we need only to consider the following two subcases.

(c. 3. 2. 1) $\{G_{m_{2l+1}},\cdots,G_{m_{n_3-2k}}\}=\{\{1,2,6\},\{1,3,6\}\}$.

(c. 3. 2. 2) $\{G_{m_{2l+1}},\cdots,G_{m_{n_3-2k}}\}=\{\{1,2,6\},\{1,3,6\},\{2,3,6\}\}$.

As to (c. 3. 2. 1), all the 4 elements in $\{1,2,3,6\}$ belong to at least two sets among the 3 sets in $\{\{1,2,6\},\{1,3,6\},\{1,2,3,4,5\}\}$. Hence in this case all the 3 elements in $\{1,2,3,6\}\cap\{1,2,3,4,5\}$ (i. e. $\{1,2,3\}$) belong to at least half of the sets in $F$.

As to (c. 3. 2. 2), we can easily know that all the 3 elements in $\{1,2,3\}$ belong to at least half of the sets in $F$.

(1. 2. 2) $n_4$ is an odd number and there exists a permutation $(i_1,\cdots,i_{n_4})$ of $(1,\cdots,n_4)$ such that $H_{i_1}\cup H_{i_2}=\cdots=H_{i_{n_4-2}}\cup H_{i_{n_4-1}}=M_6$. Then all the 6 elements in $M_6$ belong to at least half of the sets in $\{H_{i_1},\cdots,H_{i_{n_4-1}}\}$.

(1. 2. 2. 1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$. If $H_{i_{n_4}}\subset I_1$, then all the 4 elements in $H_{i_{n_4}}$ belong to at least two sets among the three sets in $\{G_1,H_{i_{n_4}},I_1\}$ and thus all the 4 elements in $H_{i_{n_4}}$ belong to at least half of the sets in $F$. If $H_{i_{n_4}}\not\subseteq I_1$, then $H_{i_{n_4}}\cup I_1=M_6$ and thus in this case all the 3 elements in $G_1$ belong to at least half of the sets in $F$.

(1. 2. 2. 2) $n_3\geqslant 2$. Now for any $i,j=1,\cdots,n_3,i\neq j$, we have $G_i\cup G_j=M_6$ or $G_i\cup G_j=I_1=\{1,2,3,4,5\}$ or $|G_i\cup G_j|=4$.

(a) $n_3$ is an even number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-1}}\cup G_{j_{n_3}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\boldsymbol{G}_3$. Hence in this case all the elements in $H_{i_{n_4}}\cup I_1$ belong to at least half of the sets in $F$.

(b) $n_3$ is an odd number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-2}}\cup G_{j_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{n_3-1}}\}$. Now if $H_{i_{n_4}}\subset I_1$, then all the 4 elements in $H_{i_{n_4}}$ belong to at least two sets among the three sets in $\{G_{j_{n_3}},H_{i_{n_4}},I_1\}$ and thus all the 4 elements in $H_{i_{n_4}}$ belong to at least half of the sets in $F$. If $H_{i_{n_4}}\not\subseteq I_1$, then $H_{i_{n_4}}\cup I_1=M_6$ and thus in this case all the 3 elements in $G_{j_{n_3}}$ belong to at least half of the sets in $F$.

(c) We can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{j_1},\cdots,G_{j_{2k}}\}$ and $\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}$, where $\{j_1,\cdots,j_{n_3}\}=\{1,\cdots,n_3\}$, $n_3-2k\geqslant 2$, and

(i) $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{2k-1}}\cup G_{j_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{j_{2k+1},\cdots,j_{n_3}\}$, $G_i\cup G_j=I_1=\{1,2,3,4,5\}$ or $|G_i\cup G_j|=4$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{2k}}\}$.

(c. 1) $n_3-2k$ is an even number and there exists a permutation $(m_1,\cdots,m_{n_3-2k})$ of $(j_{2k+1},\cdots,j_{n_3})$ such that $G_{m_1}\cup G_{m_2}=\cdots=G_{m_{n_3-2k-1}}\cup G_{m_{n_3-2k}}=\{1,2,3,4,5\}$, which together with the fact that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least one of the two sets $H_{i_{n_4}}$ and $\{1,2,3,4,5\}$, implies that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(c. 2) $n_3-2k$ is an odd number and there exists a permutation $(m_1,\cdots,m_{n_3-2k})$ of $(j_{2k+1},\cdots,j_{n_3})$ such that $G_{m_1}\cup G_{m_2}=\cdots=G_{m_{n_3-2k-2}}\cup G_{m_{n_3-2k-1}}=\{1,2,3,4,5\}$. Then all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $\{G_{m_1},\cdots,G_{m_{n_3-2k-1}}\}$.

(c. 2. 1) If $H_{i_{n_4}}\subset\{1,2,3,4,5\}$, then all the 4 elements in $H_{i_{n_4}}$ belong to at least two sets among the three sets in $\{G_{m_{n_3-2k}},H_{i_{n_4}},\{1,2,3,4,5\}\}$. Hence in this case, all the 4 elements in $H_{i_{n_4}}$ belong to at least half of the sets in $F$.

(c. 2. 2) If $H_{i_{n_4}}\not\subset\{1,2,3,4,5\}$, then $H_{i_{n_4}}\cup\{1,2,3,4,5\}=M_6$. Without loss of generality, we assume that $H_{i_{n_4}}=\{1,2,3,6\}$.

(c. 2. 2. 1) $G_{m_{n_3-2k}}\subset\{1,2,3,4,5\}$. Then all the 3 elements in $G_{m_{n_3-2k}}$ belong to at least two sets among the three sets in $\{G_{m_{n_3-2k}},H_{i_{n_4}},\{1,2,3,4,5\}\}$ and thus belong to at least half of the sets in $F$.

(c. 2. 2. 2) $G_{m_{n_3-2k}}\not\subset\{1,2,3,4,5\}$. Then $G_{m_{n_3-2k}}\cup\{1,2,3,4,5\}=M_6$. Hence all the 6 elements in $M_6$ belong to at least one of the two sets $G_{m_{n_3-2k}}$ and $\{1,2,3,4,5\}$ and thus all the 4 elements in $H_{i_{n_4}}$ belong to at least two sets among the three sets in $\{G_{m_{n_3-2k}},H_{i_{n_4}},\{1,2,3,4,5\}\}$. Hence in this case all the elements in $H_{i_{n_4}}\cap\{1,2,3,4,5\}$ (i. e. $\{1,2,3\}$) belong to at least half of the sets in $F$.

(c. 3) We can decompose $\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}$ into two disjoint parts $\{G_{m_1},\cdots,G_{m_{2l}}\}$ and $\{G_{m_{2l+1}},\cdots,G_{m_{n_3-2k}}\}$, where $\{m_1,\cdots,m_{n_3-2k}\}=\{j_{2k+1},\cdots,j_{n_3}\}$, $n_3-2k-2l\geqslant 2$, and

(iii) $G_{m_1}\cup G_{m_2}=\cdots=G_{m_{2l-1}}\cup G_{m_{2l}}=\{1,2,3,4,5\}$;

(iv) for any two different indexes $\{i,j\}$ from $\{m_{2l+1},\cdots,m_{n_3-2k}\}$, $G_i\cup G_j\in G_4$.

Then all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $\{G_{m_1},\cdots,G_{m_{2l}}\}$.

(c. 3. 1) $H_{i_{n_4}}\subset\{1,2,3,4,5\}$.

(c. 3. 1. 1)

There exists $t\in\{m_{2l+1},\cdots,m_{n_3-2k}\}$ such that $G_t\subset\{1,2,3,4,5\}$. Without loss of generality, we assume that $G_{m_{2l+1}}=\{1,2,3\}$. Then for any $t\in\{m_{2l+2},\cdots,m_{n_3-2k}\}$, we have $G_t\in\boldsymbol{H}_4\cup\boldsymbol{H}_5\cup\boldsymbol{H}_6$, where

$\boldsymbol{H}_4=\{\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$,

$\boldsymbol{H}_5=\{\{1,2,5\},\{1,3,5\},\{2,3,5\}\}$,

$\boldsymbol{H}_6=\{\{1,2,6\},\{1,3,6\},\{2,3,6\}\}$.

For simplicity, define $\boldsymbol{H}:=\{G_{m_{2l+1}},\cdots,G_{m_{n_3-2k}}\}$. We have the following 7 cases.

(c. 3. 1. 1. 1) $\boldsymbol{H}\cap\boldsymbol{H}_4\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_5=\boldsymbol{H}\cap\boldsymbol{H}_6=\varnothing$.

(c. 3. 1. 1. 2) $\boldsymbol{H}\cap\boldsymbol{H}_5\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_4=\boldsymbol{H}\cap\boldsymbol{H}_6=\varnothing$.

(c. 3. 1. 1. 3) $\boldsymbol{H}\cap\boldsymbol{H}_6\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_4=\boldsymbol{H}\cap\boldsymbol{H}_5=\varnothing$.

(c. 3. 1. 1. 4) $\boldsymbol{H}\cap\boldsymbol{H}_4\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_5\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_6=\varnothing$.

(c. 3. 1. 1. 5) $\boldsymbol{H}\cap\boldsymbol{H}_4\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_6\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_5=\varnothing$.

(c. 3. 1. 1. 6) $\boldsymbol{H}\cap\boldsymbol{H}_5\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_6\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_4=\varnothing$.

(c. 3. 1. 1. 7) $\boldsymbol{H}\cap\boldsymbol{H}_4\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_5\neq\varnothing,\boldsymbol{H}\cap\boldsymbol{H}_6\neq\varnothing$.

By the analysis in (1. 2. 1. 2)(c. 3. 1), we need only to consider the first four cases (c. 3. 1. 1. 1)-(c. 3. 1. 1. 4).

As to (c. 3. 1. 1. 1), by Lemma 1. 1, we know that all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $\boldsymbol{H}$. Since $H_{i_{n_4}}\subset\{1,2,3,4,5\}$, we know that $|H_{i_{n_4}}\cap\{1,2,3,4\}|\geqslant 3$.

Hence in this case, all the elements in $H_{i_{n_4}} \cap \{1,2,3,4\}$ belong to at least half of the sets in $F$.

As to (c.3.1.1.2), by following the analysis in (c.3.1.1.1), we get that $|H_{i_{n_4}} \cap \{1,2,3,5\}| \geqslant 3$ and all the elements in $H_{i_{n_4}} \cap \{1,2,3,5\}$ belong to at least half of the sets in $F$.

As to (c.3.1.1.3), by Lemma 1.1, we know that all the 4 elements in $\{1,2,3,6\}$ belong to at least half of the sets in $\boldsymbol{H}$. We have the following 3 subcases.

(c.3.1.1.3-1) $\boldsymbol{H} \cap \boldsymbol{H}_6 = 1$. Take $\boldsymbol{H} \cap \boldsymbol{H}_6 = \{\{1,2,6\}\}$ for example. By $H_{i_{n_4}} \subset \{1,2,3,4,5\}$, we know that $H_{i_{n_4}} \in \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\}$. If $H_{i_{n_4}} = \{1,2,3,4\}$, then $\{1,2,6\} \cup \{1,2,3,4\} = \{1,2,3,4,6\} \in \boldsymbol{G}_5 = \{\{1,2,3,4,5\}\}$. It is impossible. Similarly, if $H_{i_{n_4}} \in \{\{1,2,3,5\},\{1,2,4,5\}\}$, it is impossible. If $H_{i_{n_4}} \in \{\{1,3,4,5\},\{2,3,4,5\}\}$, then $H_{i_{n_4}} \cup \{1,2,6\} = M_6$. Now all the 3 elements in $\{1,2,3\}$ belong to at least half of the sets in $F$.

(c.3.1.1.3-2) $\boldsymbol{H} \cap \boldsymbol{H}_6 = 2$. Take $\boldsymbol{H} \cap \boldsymbol{H}_6 = \{\{1,2,6\},\{1,3,6\}\}$ for example. By following analysis in (c.3.1.1.3-1), we get that all the 3 elements in $\{1,2,3\}$ belong to at least half of the sets in $F$.

(c.3.1.1.3-3) $\boldsymbol{H} \cap \boldsymbol{H}_6 = 3$. Now $\boldsymbol{H} \cap \boldsymbol{H}_6 = \{\{1,2,6\},\{1,3,6\},\{2,3,6\}\}$. In this case, by $G_5 = \{\{1,2,3,4,5\}\}$, we must have $H_{i_{n_4}} = \{1,3,4,5\}$. It is easy to check that in this case all the 3 elements in $\{1,2,3\}$ belong to at least half of the sets in $F$.

(c.3.1.2) For any $t \in \{m_{2l+1},\cdots,m_{n_3-2k}\}$, $G_t \not\subseteq \{1,2,3,4,5\}$. Without loss of generality, we assume that $G_{m_{2l+1}} = \{1,2,6\}$. Then by (iv), we know that for any $t = m_{2l+2},\cdots,m_{n_3-2k}$, we have $G_t \in \{\{1,3,6\},\{1,4,6\},\{1,5,6\},\{2,3,6\},\{2,4,6\},\{2,5,6\}\}$. Without loss of generality, we assume that $\{1,3,6\} \in \{G_{m_{2l+2}},\cdots,G_{m_{n_3-2k}}\}$. Then by $G_5 = \{\{1,2,3,4,5\}\}$, we need only to consider the following two subcases.

(c.3.1.1.2-1) $\{G_{m_{2l+1}},\cdots,G_{m_{n_3-2k}}\} = \{\{1,2,6\},\{1,3,6\}\}$.

(c.3.1.1.2-2) $\{G_{m_{2l+1}},\cdots,G_{m_{n_3-2k}}\} = \{\{1,2,6\},\{1,3,6\},\{2,3,6\}\}$.

As to (c.3.1.1.2-1), we know that all the 3 elements in $\{1,2,3\}$ belong to at least 2 sets among the 4 sets in $\{\{1,2,6\},\{1,3,6\},H_{i_{n_4}},I_1\}$. Hence in this case, all the 3 elements in $\{1,2,3\}$ belong to at least half of the sets in $F$.

As to (c.3.1.1.2-2), we can easily know that all the 3 elements in $\{1,2,3\}$ belong to at least 3 sets among the 5 sets in $\{\{1,2,6\},\{1,3,6\},\{2,3,6\},H_{i_{n_4}},I_1\}$. Hence in this case, all the 3 elements in $\{1,2,3\}$ belong to at least half of the sets in $F$.

(c.3.2) $H_{i_{n_4}} \not\subseteq \{1,2,3,4,5\}$. Then $H_{i_{n_4}} \cup \{1,2,3,4,5\} = M_6$. By following the analysis in (c.3.1), we can show that there exist 3 elements in $M_6$ which belong to at least half of the sets in $F$. We omit the details.

(1.2.3) We can decompose $\boldsymbol{G}_4$ into two disjoint parts $\{H_{i_1},\cdots,H_{i_{2k}}\}$ and $\{H_{i_{2k+1}},\cdots,H_{i_{n_4}}\}$, where $\{i_1,\cdots,i_{n_4}\} = \{1,\cdots,n_4\}$, $n_4 - 2k \geqslant 2$, and

(i) $H_{i_1} \cup H_{i_2} = \cdots = H_{i_{2k-1}} \cup H_{i_{2k}} = M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{i_{2k+1},\cdots,i_{n_4}\}$, we have $H_i \cup H_j = I_1 = \{1,2,3,4,5\}$.

Then all the 6 elements in $M_6$ belong to at least half of the sets in $\{H_{i_1},\cdots,H_{i_{2k}}\}$. By Lemma 1.1, we know that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least $n_3 - 2k - 1$ set(s) in $\{H_{i_{2k+1}},\cdots,H_{i_{n_4}}\}$ and thus belong to at least half of the sets in $\{H_{i_{2k+1}},\cdots,H_{i_{n_4}}\}$.

(1.2.3.1) $n_3 = 1$. Now $\boldsymbol{G}_3 = \{G_1\}$. Note that all the 5 elements in $I_1 = \{1,2,3,4,5\}$ belong to at least one of the two sets $G_1$ and $I_1$. Then we obtain that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(1.2.3.2) $n_3 \geqslant 2$. Now for any $i,j = 1,\cdots,n_3, i \neq j$, we have $G_i \cup G_j = M_6$ or $G_i \cup G_j = I_1 = \{1,2,3,4,5\}$ or $|G_i \cup G_j| = 4$.

(a) $n_3$ is an even number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1} \cup G_{j_2} = \cdots = G_{j_{n_3-1}} \cup G_{j_{n_3}} = M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\boldsymbol{G}_3$.

Hence in this case all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(b) $n_3$ is an odd number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-2}}\cup G_{j_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{n_3-1}}\}$. Note that all the 5 elements in $I_1=\{1,2,3,4,5\}$ belong to at least one of the two sets $G_{j_{n_3}}$ and $I_1$. Then we obtain that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(c) We can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{j_1},\cdots,G_{j_{2k}}\}$ and $\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}$, where $\{j_1,\cdots,j_{n_3}\}=\{1,\cdots,n_3\}$, $n_3-2k\geqslant 2$, and

(i) $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{2k-1}}\cup G_{j_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{j_{2k+1},\cdots,j_{n_3}\}$, $G_i\cup G_j=I_1=\{1,2,3,4,5\}$ or $|G_i\cup G_j|=4$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{2k}}\}$.

(c. 1) $n_3-2k$ is an even number and there exists a permutation $(m_1,\cdots,m_{n_3-2k})$ of $(j_{2k+1},\cdots,j_{n_3})$ such that $G_{m_1}\cup G_{m_2}=\cdots=G_{m_{n_3-2k-1}}\cup G_{m_{n_3-2k}}=\{1,2,3,4,5\}$. In this case, we get that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(c. 2) $n_3-2k$ is an odd number and there exists a permutation $(m_1,\cdots,m_{n_3-2k})$ of $(j_{2k+1},\cdots,j_{n_3})$ such that $G_{m_1}\cup G_{m_2}=\cdots=G_{m_{n_3-2k-2}}\cup G_{m_{n_3-2k-1}}=\{1,2,3,4,5\}$. Then all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $\{G_{m_1},\cdots,G_{m_{n_3-2k-1}}\}$. Note that all the 5 elements in $I_1=\{1,2,3,4,5\}$ belong to at least one of the two sets $G_{m_{n_3-2k}}$ and $I_1$. Then we obtain that all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $F$.

(c. 3) We can decompose $\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}$ into two disjoint parts $\{G_{m_1},\cdots,G_{m_{2l}}\}$ and $\{G_{m_{2l+1}},\cdots,G_{m_{n_3-2k}}\}$, where $\{m_1,\cdots,m_{n_3-2k}\}=\{j_{2k+1},\cdots,j_{n_3}\}$, $n_3-2k-2l\geqslant 2$, and

(iii) $G_{m_1}\cup G_{m_2}=\cdots=G_{m_{2l-1}}\cup G_{m_{2l}}=\{1,2,3,4,5\}$;

(iv) for any two different indexes $\{i,j\}$ from $\{m_{2l+1},\cdots,m_{n_3-2k}\}$, $G_i\cup G_j\in G_4$.

Then all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $\{G_{m_1},\cdots,G_{m_{2l}}\}$. By following the analysis in (1. 2. 1. 2)(c. 3), we can get that there exist 3 elements in $M_6$ which belong to at least half of the sets in $F$.

(2) $n_5\geqslant 2$. By Lemma 1. 1, we know that all the 6 elements in $M_6$ belong to at least $n_5-1$ set(s) in $\boldsymbol{G}_5$ and thus belong to at least half of the sets in $\boldsymbol{G}_5$. Hence it is enough to show that there exist at least 3 elements in $M_6$ which belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

(2. 1) $n_4=1$. Now $\boldsymbol{G}_4=\{H_1\}$. Without loss of generality, we assume that $H_1=\{1,2,3,4\}$.

(2. 1. 1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$. In this case, all the elements in $\boldsymbol{G}_1\cup\boldsymbol{H}_1$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

(2. 1. 2) $n_3\geqslant 2$. Now for any $i,j=1,\cdots,n_3$, $i\neq j$, we have $G_i\cup G_j=M_6$ or $G_i\cup G_j\in\boldsymbol{G}_5$ or $G_i\cup G_j=\boldsymbol{H}_1$.

(2. 1. 2. 1) $n_3$ is an even number and there exists a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-1}}\cup G_{i_{n_3}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $G_3$. Hence in this case all the 4 elements in $I_1$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

(2. 1. 2. 2) $n_3$ is an odd number and there exists a permutation $(i_1,\cdots,i_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{n_3-2}}\cup G_{i_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{n_3-1}}\}$. In this case, all the elements in $G_{i_3}\cup I_1$ belong to at least half of the sets in $G_3\cup G_4$.

(2. 1. 2. 3) We can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{i_1},\cdots,G_{i_{2k}}\}$ and $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$, where $\{i_1,\cdots,i_{n_3}\}=\{1,\cdots,n_3\}$, $n_3-2k\geqslant 2$, and

(i) $G_{i_1}\cup G_{i_2}=\cdots=G_{i_{2k-1}}\cup G_{i_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{i_{2k+1},\cdots,i_{n_3}\}$, $G_i\cup G_j=H_1=\{1,2,3,4\}$ or $G_i\cup G_j\in G_5$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{i_1},\cdots,G_{i_{2k}}\}$.

(a) $n_3-2k$ is an even number and there exists a permutation $(j_1,\cdots,j_{n_3-2k})$ of $(i_{2k+1},\cdots,i_{n_3})$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-2k-1}}\cup G_{j_{n_3-2k}}=H_1=\{1,2,3,4\}$. Then all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$ and thus all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

(b) $n_3-2k$ is an odd number and there exists a permutation $(j_1,\cdots,j_{n_3-2k})$ of $(i_{2k+1},\cdots,i_{n_3})$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-2k-2}}\cup G_{j_{n_3-2k-1}}=H_1=\{1,2,3,4\}$. Then all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $\{G_{j_1},\cdots,G_{j_{n_3-2k-1}}\}$. Note that all the the 4 elements in $\{1,2,3,4\}$ belong to at least one of the two sets $G_{j_{n_3-2k}}$ and $\{1,2,3,4\}$. Then we get that all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

(c) We can decompose $\{G_{i_{2k+1}},\cdots,G_{i_{n_3}}\}$ into two disjoint two parts $\{G_{j_1},\cdots,G_{j_{2l}}\}$ and $\{G_{j_{2l+1}},\cdots,G_{j_{n_3-2k}}\}$, where $\{j_1,\cdots,j_{n_3-2k}\}=\{i_{2k+1},\cdots,i_{n_3}\}$, $n_3-2k-2l\geqslant 2$, and

(iii) $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{2l-1}}\cup G_{j_{2l}}=\{1,2,3,4\}$;

(iv) for any two different indexes $\{i,j\}$ from $\{j_{2l+1},\cdots,j_{n_3-2k}\}$, we have $G_i\cup G_j\in G_5$.

Then all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $\{G_{j_1},\cdots,G_{j_{2l}}\}$. For simplicity, define $\boldsymbol{H}:=\{G_{j_{2l+1}},\cdots,G_{j_{n_3-2k}}\}$. We have the following two subcases:

(c.1) There exists $m\in\{j_{2l+1},\cdots,j_{n_3-2k}\}$ such that $G_m\subset\{1,2,3,4\}$. Without loss of generality, we assume that $G_{j_{2l+1}}=\{1,2,3\}$. Then for any $m=j_{2l+2},\cdots,j_{n_3-2k}$, we have $G_m\in H_{4,5}\cup H_{4,6}\cup H_{5,6}$, where

$$H_{4,5}=\{\{1,4,5\},\{2,4,5\},\{3,4,5\}\},$$
$$H_{4,6}=\{\{1,4,6\},\{2,4,6\},\{3,4,6\}\},$$
$$H_{5,6}=\{\{1,5,6\},\{2,5,6\},\{3,5,6\}\}.$$

Since $|\{1,4,5\}\cup\{2,4,5\}|=|\{1,4,5\}\cup\{3,4,5\}|=|\{2,4,5\}\cup\{3,4,5\}|=4$, we get that $|\boldsymbol{H}\cap H_{4,5}|\leqslant 1$. Similarly, we have $|\boldsymbol{H}\cap H_{4,6}|\leqslant 1$, $|\boldsymbol{H}\cap H_{5,6}|\leqslant 1$. Hence we need only to consider the following 7 cases.

(c.1.1) $|\boldsymbol{H}\cap H_{4,5}|=1$, $|\boldsymbol{H}\cap H_{4,6}|=|\boldsymbol{H}\cap H_{5,6}|=0$.

(c.1.2) $|\boldsymbol{H}\cap H_{4,6}|=1$, $|\boldsymbol{H}\cap H_{4,5}|=|\boldsymbol{H}\cap H_{5,6}|=0$.

(c.1.3) $|\boldsymbol{H}\cap H_{5,6}|=1$, $|\boldsymbol{H}\cap H_{4,5}|=|\boldsymbol{H}\cap H_{4,6}|=0$.

(c.1.4) $|\boldsymbol{H}\cap H_{4,5}|=|\boldsymbol{H}\cap H_{4,6}|=1$, $|\boldsymbol{H}\cap H_{5,6}|=0$.

(c.1.5) $|\boldsymbol{H}\cap H_{4,5}|=|\boldsymbol{H}\cap H_{5,6}|=1$, $|\boldsymbol{H}\cap H_{4,6}|=0$.

(c.1.6) $|\boldsymbol{H}\cap H_{4,6}|=|\boldsymbol{H}\cap H_{5,6}|=1$, $|\boldsymbol{H}\cap H_{4,5}|=0$.

(c.1.7) $|\boldsymbol{H}\cap H_{4,5}|=|\boldsymbol{H}\cap H_{4,6}|=|\boldsymbol{H}\cap H_{5,6}|=1$.

As to (c.1.1), take $\boldsymbol{H}\cap H_{4,5}=\{\{1,4,5\}\}$ for example. Now $\boldsymbol{H}=\{\{1,2,3\},\{1,4,5\}\}$. Note that all the 4 elements in $\{1,2,3,4\}$ belong to at least two sets among the three sets in $\{\{1,2,3\},\{1,4,5\},\{1,2,3,4\}\}$. Then we get that in this case, all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

As to (c.1.2), by following the analysis to (c.1.1), we get that all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

As to (c.1.3), by following the analysis to (c.1.1), we get that all the 3 elements in $\{1,2,3\}$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

As to (c.1.4), take $\boldsymbol{H}\cap(H_{4,5}\cup H_{4,6})=\{\{1,4,5\},\{2,4,6\}\}$ for example. Now $H=\{\{1,2,3\},\{1,4,5\},\{2,4,6\}\}$. Note that all the 4 elements in $\{1,2,3,4\}$ belong to at least two sets among the four sets in $\{\{1,2,3\},\{1,4,5\},\{2,4,6\},\{1,2,3,4\}\}$. Then we get that in this case, all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

As to (c.1.5), take $\boldsymbol{H}\cap(H_{4,5}\cup H_{4,6})=\{\{1,4,5\},\{2,5,6\}\}$ for example. Now $\boldsymbol{H}=\{\{1,2,3\},\{1,4,5\},\{2,5,6\}\}$. Note that all the 4 elements in $\{1,2,3,4\}$ belong to at least two sets among the four sets in $\{\{1,2,3\},\{1,4,5\},\{2,5,6\},\{1,2,3,4\}\}$. Then we get that in this case, all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

As to (c.1.6), take $\boldsymbol{H}\cap(H_{4,6}\cup H_{5,6})=\{\{1,4,6\},\{2,5,6\}\}$ for example. Now $\boldsymbol{H}=\{\{1,2,3\},\{1,4,6\},\{2,5,6\}\}$. Note that all the 4 elements in $\{1,2,3,4\}$ belong to at least two sets among the four sets in $\{\{1,2,3\},\{1,4,6\},\{2,5,6\},\{1,2,3,4\}\}$. Then we get that in this case, all the 4 elements in $\{1,2,3,4\}$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

As to (c.1.7), take $\boldsymbol{H}\cap H_{4,5}=\{\{1,4,5\},\{2,4,6\},\{3,5,6\}\}$ for example. Now $\boldsymbol{H}=\{\{1,2, 3},{1,4,5},{2,4,6},{3,5,6}}. Note that all the 4 elements in {1,2,3,4} belong to at least 3 sets among the 5 sets in {{1,2,3},{1,4,5},{2,4,6},{3,5,6},{1,2,3,4}}. Then we get that in this case, all the 4 elements in {1,2,3,4} belong to at least half of the sets in $\boldsymbol{G}_3 \cup \boldsymbol{G}_4$.

(c. 2) For any $m \in \{j_{2l+1}, \cdots, j_{n_3-2k}\}$, $G_m \not\subseteq \{1,2,3,4\}$. Then $\boldsymbol{H} \subset \boldsymbol{H}_5 \cup \boldsymbol{H}_6 \cup \boldsymbol{H}_{5,6}$, where

$$\boldsymbol{H}_5 = \{\{\{1,2,5\},\{1,3,5\},\{1,4,5\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\},$$
$$\boldsymbol{H}_6 = \{\{\{1,2,6\},\{1,3,6\},\{1,4,6\},\{2,3,6\},\{2,4,6\},\{3,4,6\}\},$$
$$\boldsymbol{H}_{5,6} = \{\{1,5,6\},\{2,5,6\},\{3,5,6\},\{4,5,6\}\}.$$

By (iv), we can easily get that

$$|\boldsymbol{H} \cap \boldsymbol{H}_5| \leqslant 2, |\boldsymbol{H} \cap \boldsymbol{H}_6| \leqslant 2, |\boldsymbol{H} \cap \boldsymbol{H}_{5,6}| \leqslant 1.$$

(c. 2. 1) $\boldsymbol{H} \cap \boldsymbol{H}_{5,6} = \Phi$. Without loss of generality, we assume that $\{1,2,5\} \in \boldsymbol{H}$ and $G_{j_{2l+1}} = \{1,2,5\}$. Then by (iv), we know that for any $m \in \{j_{2l+2}, \cdots, j_{n_3-2k}\}$, $G_m \in \{\{3,4,5\},\{1,3,6\},\{1,4,6\},\{2,3,6\},\{2,4,6\}\}$. By (iv) again, we need only to consider the following 14 cases.

(c. 2. 1. 1) $H = \{\{1,2,5\},\{3,4,5\}\}$

(c. 2. 1. 2) $H = \{\{1,2,5\},\{1,3,6\}\}$

(c. 2. 1. 3) $H = \{\{1,2,5\},\{1,4,6\}\}$

(c. 2. 1. 4) $H = \{\{1,2,5\},\{2,3,6\}\}$

(c. 2. 1. 5) $H = \{\{1,2,5\},\{2,4,6\}\}$

(c. 2. 1. 6) $H = \{\{1,2,5\},\{3,4,5\},\{1,3,6\}\}$

(c. 2. 1. 7) $H = \{\{1,2,5\},\{3,4,5\},\{1,4,6\}\}$

(c. 2. 1. 8) $H = \{\{1,2,5\},\{3,4,5\},\{2,3,6\}\}$

(c. 2. 1. 9) $H = \{\{1,2,5\},\{3,4,5\},\{2,4,6\}\}$

(c. 2. 1. 10) $H = \{\{1,2,5\},\{1,3,6\},\{2,4,6\}\}$

(c. 2. 1. 11) $H = \{\{1,2,5\},\{1,4,6\},\{2,3,6\}\}$

(c. 2. 1. 12) $H = \{\{1,2,5\},\{1,3,6\},\{2,4,6\}\}$

(c. 2. 1. 13) $H = \{\{1,2,5\},\{3,4,5\},\{1,3,6\},\{2,4,6\}\}$

(c. 2. 1. 14) $H = \{\{1,2,5\},\{3,4,5\},\{1,4,6\},\{2,3,6\}\}$

As to (c. 2. 1. 1), all the 4 elements in {1,2,3,4} belong to at least two sets among the three sets in {{1,2,5},{3,4,5},{1,2,3,4}}. Then we get that all the 4 elements in {1,2,3,4} belong to at least half of the sets in $\boldsymbol{G}_3 \cup \boldsymbol{G}_4$.

As to (c. 2. 1. 2) and (c. 2. 1. 4), all the 3 elements in {1,2,3} belong to at least half of the sets in $\boldsymbol{G}_3 \cup \boldsymbol{G}_4$.

As to (c. 2. 1. 3) and (c. 2. 1. 5), all the 3 elements in {1,2,4} belong to at least half of the sets in $\boldsymbol{G}_3 \cup \boldsymbol{G}_4$.

As to (c. 2. 1. 6), all the 4 elements in {1,2,3,4} belong to at least two sets among the four sets in {{1,2,5},{3,4,5},{1,3,6},{1,2,3,4}}. Then we get that all the 4 elements in {1,2,3,4} belong to at least half of the sets in $\boldsymbol{G}_3 \cup \boldsymbol{G}_4$.

As to (c. 2. 1. 7)-(c. 2. 1. 12), it is easy to check that all the 4 elements in {1,2,3,4} belong to at least two sets among the four sets in $\boldsymbol{H} \cup \{\{1,2,3,4\}\}$ and thus belong to at least half of the sets in $\boldsymbol{G}_3 \cup \boldsymbol{G}_4$.

As to (c. 2. 1. 13) and (c. 2. 1. 14), it is easy to check that all the 4 elements in {1,2,3,4} belong to at least three sets among the 5 sets in $\boldsymbol{H} \cup \{\{1,2,3,4\}\}$ and thus belong to at least half of the sets in $\boldsymbol{G}_3 \cup \boldsymbol{G}_4$.

(c. 2. 2) $\boldsymbol{H} \cap \boldsymbol{H}_{5,6} \neq \Phi$. Without loss of generality, we assume that $\{1,5,6\} \in \boldsymbol{H}$ and $G_{j_{2l+1}} = \{1,5,6\}$. Then by (iv), we know that for any $m \in \{j_{2l+2}, \cdots, j_{n_3-2k}\}$, $G_m \in \{\{2,3,5\},\{2,4,5\},\{3,4,5\},\{2,3,6\},\{2,4,6\},\{3,4,6\}\}$. By (iv) again, we need only to consider the following 3 cases.

(c. 2. 2. 1) $|\boldsymbol{H} \cap \{\{2,3,5\},\{2,4,5\},\{3,4,5\}\}| = 1$, $|\boldsymbol{H} \cap \{\{2,3,6\},\{2,4,6\},\{3,4,6\}\}| = 0$.

(c. 2. 2. 2) $|\boldsymbol{H} \cap \{\{2,3,5\},\{2,4,5\},\{3,4,5\}\}| = 0$, $|\boldsymbol{H} \cap \{\{2,3,6\},\{2,4,6\},\{3,4,6\}\}| = 1$.

(c. 2. 2. 3) $|\boldsymbol{H} \cap \{\{2,3,5\},\{2,4,5\},\{3,4,5\}\}| = 1$, $|\boldsymbol{H} \cap \{\{2,3,6\},\{2,4,6\},\{3,4,6\}\}| = 1$.

As to (c. 2. 2. 1), take $\boldsymbol{H} \cap \{\{2,3,5\},\{2,4,5\},\{3,4,5\}\} = \{\{2,3,5\}\}$ for example. Now $\boldsymbol{H} = \{\{1,5,6\},\{2,3,5\}\}$. Now all the 3 elements in {1,2,3} belong to at least two sets among the three sets in $\boldsymbol{H} \cup \{\{1,2,3,4\}\}$ and thus belong to at least half of the sets in $\boldsymbol{G}_3 \cup \boldsymbol{G}_4$.

As to (c. 2. 2. 2), take $\boldsymbol{H} \cap \{\{2,3,6\},\{2,4,6\},\{3,4,6\}\} = \{\{2,3,6\}\}$ for example. Now $\boldsymbol{H} =$

$\{\{1,5,6\},\{2,3,6\}\}$. Now all the 3 elements in $\{1,2,3\}$ belong to at least two sets among the three sets in $\boldsymbol{H}\cup\{\{1,2,3,4\}\}$ and thus belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

As to (c. 2. 2. 3), take $\boldsymbol{H}=\{\{1,2,5\},\{2,3,5\},\{2,4,6\}\}$ for example. Now all the 4 elements in $\{1,2,3,4\}$ belong to at least two sets among the four sets in $\boldsymbol{H}\cup\{\{1,2,3,4\}\}$ and thus belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_4$.

(2. 2) $n_4\geqslant 2$. For any $i,j=1,\cdots,n_4,i\neq j$, we have $H_i\cup H_j=M_6$ or $H_i\cup H_j\in\boldsymbol{G}_5$.

(2. 2. 1) $n_4$ is an even number and there exists a permutation $(i_1,\cdots,i_{n_4})$ of $(1,\cdots,n_4)$ such that $H_{i_1}\cup H_{i_2}=\cdots=H_{i_{n_4-1}}\cup H_{i_{n_4}}=M_6$. Then all the 6 elements in $M_6$ belong to at least half of the sets in $\boldsymbol{G}_4$. Hence it is enough to show that there exist 3 elements in $M_6$ which belong to at least half of the sets in $\boldsymbol{G}_3$ or $\boldsymbol{G}_3\cup\boldsymbol{G}_5$.

(2. 2. 1. 1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$, and all the 3 elements in $G_1$ satisfy the condition.

(2. 2. 1. 2) $n_3=2$. Now $\boldsymbol{G}_3=\{G_1,G_2\}$ and all the 3 elements in $G_1\cup G_2$ belong to at least one of the two sets in $\boldsymbol{G}_3$.

(2. 2. 1. 3) $n_3\geqslant 3$. For any $i,j=1,\cdots,n_3,i\neq j$, we have $G_i\cup G_j=M_6$ or $G_i\cup G_j\in\boldsymbol{G}_4\cup\boldsymbol{G}_5$.

(a) $n_3$ is an even number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-1}}\cup G_{j_{n_3}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\boldsymbol{G}_3$.

(b) $n_3$ is an odd number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-2}}\cup G_{j_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{n_3-1}}\}$. Hence all the 3 elements in $G_{j_{n_3}}$ belong to at least half of the sets in $\boldsymbol{G}_3$.

(c) we can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{j_1},\cdots,G_{j_{2k}}\}$ and $\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}$, where $\{j_1,\cdots,j_{n_3}\}=\{1,\cdots,n_3\},n_3-2k\geqslant 2$, and

(i) $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{2k-1}}\cup G_{j_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{j_{2k+1},\cdots,j_{n_3}\}$, $G_i\cup G_j\in\boldsymbol{G}_4\cup\boldsymbol{G}_5$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{2k}}\}$. Without loss of generality, we assume that $G_{j_{2k+1}}=\{1,2,3\}$. For any $l=j_{2k+2},\cdots,j_{n_3}$, we have $G_l\in\{\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\}\}$. For simplicity, denote $\boldsymbol{H}=\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}$. By (ii), we know that $|\boldsymbol{H}|\leqslant 10$. Then we have the following 9 cases:

(c. 1) $|\boldsymbol{H}|=2$. Now $\boldsymbol{H}=\{G_{j_{2k+1}},G_{j_{n_3}}\}$ and all the elements in $G_{j_{2k+1}}\cup G_{j_{n_3}}$ belong to at least one of the two sets in $H$ and thus belong to at least half of the sets in $G_3$.

(c. 2) $|\boldsymbol{H}|=3$. (c. 3) $|\boldsymbol{H}|=4$.

(c. 4) $|\boldsymbol{H}|=5$. (c. 5) $|\boldsymbol{H}|=6$.

(c. 6) $|\boldsymbol{H}|=7$. (c. 7) $|\boldsymbol{H}|=8$.

(c. 8) $|\boldsymbol{H}|=9$. (c. 9) $|\boldsymbol{H}|=10$.

As to the cases (c. 2)～(c. 9), we only give the proof for (c. 2), the proofs for the other cases are similar. We omit the details.

As to (c. 2), $\boldsymbol{H}=\{G_{j_{2k+1}},G_{j_{2k+2}},G_{j_{n_3}}\}$ and we have the following 4 cases:

(c. 2. 1) For any 2-element subset $\{i,j\}$ of $\{j_{2k+1},j_{2k+2},j_{n_3}\}$, $G_i\cup G_j\in G_4$ and $G_{j_{2k+1}}\cup G_{j_{2k+2}}\cup G_{j_{n_3}}\in\boldsymbol{G}_4$. Take $\mathsf{H}=\{\{1,2,3\},\{1,2,4\},\{2,3,4\}\}$ for example. Now by Lemma 1. 1. we know that all the 4 elements in $\{1,2,3,4\}$ belong to at least 2 sets in $H$ and thus belong to at least half of the sets in $\boldsymbol{G}_3$.

(c. 2. 2) For any 2-element subset $\{i,j\}$ of $\{j_{2k+1},j_{2k+2},j_{n_3}\}$, $G_i\cup G_j\in\boldsymbol{G}_4$ and $G_{j_{2k+1}}\cup G_{j_{2k+2}}\cup G_{j_{n_3}}\in\boldsymbol{G}_5$. Take $\boldsymbol{H}=\{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$ for example. Now $\{1,2,3,4,5\}\in\boldsymbol{G}_5$ and thus we have the following 3 cases.

(c. 2. 2. 1) $n_5=1$. Then $\boldsymbol{G}_5=\{\{1,2,3,4,5\}\}$. Now all the 5 elements in $\{1,2,3,4,5\}$ belong to at least two sets among the 4 sets in $\boldsymbol{H}\cup\boldsymbol{G}_5$ and thus belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_5$. (In fact, by the assumption that $n_5\geqslant 2$, we don't need consider this case. We write it here for the analysis in the following (c. 2. 2. 3)).

(c. 2. 2. 2) $n_5=2$. Take $\boldsymbol{G}_5=\{\{1,2,3,4,5\},\{1,2,3,4,6\}\}$ for example. Now it easy to check that all the 4 elements in $\{1,2,3,4\}$ belong to at least 3 sets among the 5 sets in $\boldsymbol{H}\cup\boldsymbol{G}_5$ and thus belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_5$.

(c. 2. 2. 3) $n_5\geqslant 3$. Now by Lemma 1. 1, we know that all the 6 elements in $M_6$ belong to at least $n_5-2$ set(s) in $G_5\backslash\{\{1,2,3,4,5\}\}$ and thus belong to at least half of the sets in $G_5\backslash\{\{1,2,3,4,5\}\}$. Then by (c. 2. 2. 1), we know that now all the 5 elements in $\{1,2,3,4,5\}$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_5$.

(c. 2. 3) For any 2-element subset $\{i,j\}$ of $\{j_{2k+1},j_{2k+2},j_{n_3}\}$, $G_i\cup G_j\in\boldsymbol{G}_5$. Take $\boldsymbol{H}=\{\{1,2,3\},\{1,4,5\},\{2,5,6\}\}$ for example. Now $\{\{1,2,3,4,5\},\{1,2,3,5,6\},\{1,2,4,5,6\}\}\subset\boldsymbol{G}_5$ and thus we have the following 3 cases:

(c. 2. 3. 1) $n_5=3$. Then $\boldsymbol{G}_5=\{\{1,2,3,4,5\},\{1,2,3,5,6\},\{1,2,4,5,6\}\}$. Now all the 6 elements in $M_6$ belong to at least 3 sets among the 6 sets in $\boldsymbol{H}\cup\boldsymbol{G}_5$ and thus belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_5$.

(c. 2. 3. 2) $n_5=4$. Without loss of generality, we assume that $\{1,2,3,4,6\}\in\boldsymbol{G}_5$. Then by the analysis in (c. 2. 3. 1), we know that all the 5 elements in $\{1,2,3,4,6\}$ belong to at least 4 sets among the 7 sets in $\boldsymbol{H}\cup\boldsymbol{G}_5$ and thus belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_5$.

(c. 2. 3. 3) $n_5\geqslant 5$. By Lemma 1. 1, we know that all the 6 elements in $M_6$ belong to at least $|G_5\{\{1,2,3,4,5\},\{1,2,3,5,6\},\{1,2,4,5,6\}\}|-1$ set(s) in $G_5\backslash\{\{1,2,3,4,5\},\{1,2,3,5,6\},\{1,2,4,5,6\}\}$ and thus belong to at least half of the sets in $G_5\backslash\{\{1,2,3,4,5\},\{1,2,3,5,6\},\{1,2,4,5,6\}\}$. Hence in this case, by (c. 2. 3. 1), we know that all the 6 elements in $M_6$ belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_5$.

(c. 2. 4) $|\{\{i,j\}\,|\,\{i,j\}\subset\{j_{2k+1},j_{2k+2},j_{n_3}\},G_i\cup G_j\in\boldsymbol{G}_5\}|=2$. Take $\boldsymbol{H}=\{\{1,2,3\},\{1,2,4\},\{1,5,6\}\}$ for example. Now $\{\{1,2,3,5,6\},\{1,2,4,5,6\}\}\subset\boldsymbol{G}_5$. By following the analysis in (c. 2. 2) and (c. 2. 3), we can get that there exist at least 3 elements in $M_6$ which belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_5$.

(c. 2. 5) $|\{\{i,j\}\,|\,\{i,j\}\subset\{j_{2k+1},j_{2k+2},j_{n_3}\},G_i\cup G_j\in\boldsymbol{G}_5\}|=1$. Take $\boldsymbol{H}=\{\{1,2,3\},\{1,2,5\},\{2,5,6\}\}$ for example. Now $\{1,2,3,5,6\}\in\boldsymbol{G}_5$. By following the analysis in (c. 2. 2) and (c. 2. 3), we can get that there exist at least 3 elements in $M_6$ which belong to at least half of the sets in $\boldsymbol{G}_3\cup\boldsymbol{G}_5$.

(2. 2. 2) $n_4$ is an odd number and there exists a permutation $(i_1,\cdots,i_{n_4})$ of $(1,\cdots,n_4)$ such that $H_{i_1}\cup H_{i_2}=\cdots=H_{i_{n_4-2}}\cup H_{i_{n_4-1}}=M_6$. Then all the 6 elements in $M_6$ belong to at least half of the sets in $\{H_{i_1},\cdots,H_{i_{n_4-1}}\}$. Hence it is enough to show that there exist 3 elements in $M_6$ which belong to at least half of the sets in $\{H_{i_{n_4}}\}\cup G_3$ or $\{H_{i_{n_4}}\}\cup\boldsymbol{G}_3\cup\boldsymbol{G}_5$. Without loss of generality, we assume that $H_{i_{n_4}}=\{1,2,3,4\}$.

(2. 2. 2. 1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$, and all the elements in $H_{i_{n_4}}\cup G_1$ belong to at least one of the two sets in $\{H_{i_{n_4}}\}\cup\boldsymbol{G}_3$.

(2. 2. 2. 2) $n_3\geqslant 2$. For any $i,j=1,\cdots,n_3,i\neq j$, we have $G_i\cup G_j=M_6$ or $G_i\cup G_j\in\boldsymbol{G}_4\cup\boldsymbol{G}_5$.

(a) $n_3$ is an even number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-1}}\cup G_{j_{n_3}}=M_6$. Then all the 4 elements in $H_{i_{n_4}}$ belong to at least half of the sets in $\{H_{i_{n_4}}\}\cup\boldsymbol{G}_3$.

(b) $n_3$ is an odd number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-2}}\cup G_{j_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{n_3-1}}\}$. Hence all the elements in $H_{i_{n_4}}\cup G_{j_{n_3}}$ belong to at least half of the sets in $\{H_{i_{n_4}}\}\cup\boldsymbol{G}_3$.

(c) we can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{j_1},\cdots,G_{j_{2k}}\}$ and $\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}$, where $\{j_1,\cdots,j_{n_3}\}=\{1,\cdots,n_3\},n_3-2k\geqslant 2$, and

(i) $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{2k-1}}\cup G_{j_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{j_{2k+1},\cdots,j_{n_3}\}$, $G_i\cup G_j\in\boldsymbol{G}_4\cup\boldsymbol{G}_5$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{2k}}\}$. We have the following two cases.

(c. 1) There exists $m\in\{j_{2k+1},\cdots,j_{n_3}\}$ such that $G_m\cup\{1,2,3,4\}=M_6$. Without loss of generality, we assume that $G_{j_{2k+1}}=\{1,5,6\}$. Then for any $l=j_{2k+2},\cdots,j_{n_3}$, we have $G_k\in\{\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\}\}$.

(c.2) For any $m\in\{j_{2k+1},\cdots,j_{n_3}\}$, $G_m\cup\{1,2,3,4\}\neq M_6$. Then $\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}\subset\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{3,4,5\},\{3,4,6\}\}$.

As to (c.1) and (c.2), by following the analysis in (2.2.1), we get that there exist at least 3 elements in $M_6$ which belong to at least half of the sets in $\{H_{i_{n_4}}\}\cup\boldsymbol{G}_3\cup\boldsymbol{G}_5$. We omit the details.

(1.2.3) We can decompose $\boldsymbol{G}_4$ into two disjoint parts $\{H_{i_1},\cdots,H_{i_{2k}}\}$ and $\{H_{i_{2k+1}},\cdots,H_{i_{n_4}}\}$, where $\{i_1,\cdots,i_{n_4}\}$, $n_4-2k\geqslant 2$, and

(i) $H_{i_1}\cup H_{i_2}=\cdots=H_{i_{n_4-1}}\cup H_{i_{n_4}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{i_{2k+1},\cdots,i_{n_4}\}$, $H_i\cup H_j\in\boldsymbol{G}_5$.

Then all the 6 elements in $M_6$ belong to at least half of the sets in $\{H_{i_1},\cdots,H_{i_{2k}}\}$. Without loss of generality, we assume that $H_{i_{2k+1}}=\{1,2,3,4\}$. Then by (ii), we get that for any $j=i_{2k+2},\cdots,i_{n_4}$, $H_j\in H_5\cup H_6$, where $H_5=\{\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\}$, $H_6=\{\{1,2,3,6\},\{1,2,4,6\},\{1,3,4,6\},\{2,3,4,6\}\}$. For simplicity, denote $\boldsymbol{H}=\{H_{i_{2k+1}},\cdots,H_{i_{n_4}}\}$. Then we have the following 3 cases.

(2.2.3.1) $\boldsymbol{H}\cap H_5\neq\Phi$, $\boldsymbol{H}\cap H_6=\Phi$. Now, we have the following 4 cases.

(a) $|\boldsymbol{H}\cap H_5|=1$; (b) $|\boldsymbol{H}\cap H_5|=2$,

(c) $|\boldsymbol{H}\cap H_5|=3$; (d) $|\boldsymbol{H}\cap H_5|=4$.

In the following, we only give the proof for (a). The proofs for other cases are similar. We omit the details. Without loss of generality, we assume that $\boldsymbol{H}\cap H_5=\{\{1,2,3,5\}\}$ and thus $\boldsymbol{H}=\{\{1,2,3,4\},\{1,2,3,5\}\}$.

(a.1) $n_3=1$. Now $\boldsymbol{G}_3=\{G_1\}$. Obviously, all the 3 elements in $\{1,2,3\}$ belong to at least two sets among the three sets in $\boldsymbol{H}\cup G_3$ and thus belong to at least half of the sets in $F$.

(a.2) $n_3\geqslant 2$. For any $i,j=1,\cdots,n_3,i\neq j$, we have $G_i\cup G_j=M_6$ or $G_i\cup G_j\in G_4\cup G_5$.

(a.2.1) $n_3$ is an even number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-1}}\cup G_{j_{n_3}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\boldsymbol{G}_3$. Now all the 5 elements in $\{1,2,3,4,5\}$ belong to at least one of the two sets in $\boldsymbol{H}$ and thus belong to at least half of the sets in $F$.

(a.2.2) $n_3$ is an odd number and there exists a permutation $(j_1,\cdots,j_{n_3})$ of $(1,\cdots,n_3)$ such that $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{n_3-2}}\cup G_{j_{n_3-1}}=M_6$. Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{n_3-1}}\}$. Now, all the 3 elements in $\{1,2,3\}$ belong to at least two sets among the three sets in $\boldsymbol{H}\cup\{G_{j_{n_3}}\}$ and thus belong to at least half of the sets in $F$.

(a.2.3) we can decompose $\boldsymbol{G}_3$ into two disjoint parts $\{G_{j_1},\cdots,G_{j_{2k}}\}$ and $\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}$, where $\{j_1,\cdots,j_{n_3}\}=\{1,\cdots,n_3\}$, $n_3-2k\geqslant 2$, and

(i) $G_{j_1}\cup G_{j_2}=\cdots=G_{j_{2k-1}}\cup G_{j_{2k}}=M_6$;

(ii) for any two different indexes $\{i,j\}$ from $\{j_{2k+1},\cdots,j_{n_3}\}$, $G_i\cup G_j\in\boldsymbol{G}_4\cup\boldsymbol{G}_5$.

Then all the 6 elements in $M_6$ belong to half of the sets in $\{G_{j_1},\cdots,G_{j_{2k}}\}$. Hence in this case, it is enough to show that there exist at least 3 elements in $M_6$ which belong to at least half of the sets in $\boldsymbol{H}\cup\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}$ or $\boldsymbol{H}\cup\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}\cup G_5$, where $\boldsymbol{H}=\{\{1,2,3,4\},\{1,2,3,5\}\}$.

(a.2.3.1) There exists $m\in\{j_{2k+1},\cdots,j_{n_3}\}$ such that $G_m=\{1,2,3\}=\{1,2,3,4\}\cup\{1,2,3,5\}$. Without loss of generality, we assume that $G_{j_{2k+1}}=\{1,2,3\}$.

(a.2.3.2) There exists $m\in\{j_{2k+1},\cdots,j_{n_3}\}$ such that $G_m\subset\{1,2,3,4\}$ but $G_m\not\subseteq\{1,2,3,5\}$ and for any $m\in\{j_{2k+1},\cdots,j_{n_3}\}$, $G_m\neq\{1,2,3\}$. Without loss of generality, we assume that $G_{j_{2k+1}}=\{1,2,4\}$.

(a.2.3.3) There exists $m\in\{j_{2k+1},\cdots,j_{n_3}\}$ such that $G_m\subset\{1,2,3,5\}$ but $G_m\not\subseteq\{1,2,3,4\}$ and for any $m\in\{j_{2k+1},\cdots,j_{n_3}\}$, $G_m\neq\{1,2,3\}$. Without loss of generality, we assume that $G_{j_{2k+1}}=\{1,2,5\}$.

(a.2.3.4) For any $m\in\{j_{2k+1},\cdots,j_{n_3}\}$, $G_m\not\subseteq\{1,2,3,4\}$ and $G_m\not\subseteq\{1,2,3,5\}$. Now $\boldsymbol{H}\subset\{\{1,2,6\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,6\},$

{2,4,5},{2,4,6},{2,5,6},{3,4,5},{3,4,6},{3,5,6},{4,5,6}}.

For the above 4 cases, by following the proof in (2.2.1.3), we can get that there exist at least 3 elements in $M_6$ which belong to at least half of the sets in $\boldsymbol{H}\cup\{G_{j_{2k+1}},\cdots,G_{j_{n_3}}\}\cup\boldsymbol{G}_5$ and thus belong to at least half of the sets in $F$.

(2.2.3.2) $\boldsymbol{H}\cap H_6\neq\varnothing, H\cap H_5=\varnothing$. The proof is similar to (2.2.3.1). We omit the details.

(2.2.3.3) $\boldsymbol{H}\cap H_5\neq\varnothing, H\cap H_6\neq\varnothing$. Without loss of generality, we assume that $\{1,2,3,5\}\in\boldsymbol{H}\cap H_5$, then by (ii) we know that $\boldsymbol{H}\cap H_6=\{\{1,2,3,6\}\}$ and $\boldsymbol{H}\cap H_5=\{\{1,2,3,5\}\}$. Now $\boldsymbol{H}=\{\{1,2,3,4\},\{1,2,3,5\},\{1,2,3,6\}\}$. By following the proof for (2.2.3.1), we get that there exist at least 3 elements in $M_6$ which belong to at least half of the sets in $F$.

**References**:


[1] Rival I. Graphs and order [M]. Dordrecht/Boston: Reidel, 1985.

[2] Stanley R P. Enumerative combinatorics (I) [M]. Belmont: Wadsworth & Brooks/Cole, 1986.

[3] Sarvate D G, Renaud J C. On the union-closed sets conjecture [J]. Ars Combin, 1989, 27: 149.

[4] Poonen B. Union-closed families [J]. J Comb Theory Ser A, 192, 59: 253.

[5] Bošnjak I, Marković P. The 11-element case of Frankl's conjecture [J]. Electronic J Comb, 2008, 15: R88.

[6] Lo Faro G. Union-closed sets conjecture: improved bounds [J]. J Comb Math Comb Comput, 1994, 16: 97.

[7] Roverts I, Simpson J. A note on the union-closed sets conjecture [J]. Austral J Comb, 2010, 47: 265.

[8] Bruhn H, Schaudt O. The journey of the union-closed sets conjecture [J]. Graphs Comb, 2015, 31: 2043.

[9] Gao W, Yu H. Note on the union-closed sets conjecture [J]. Ars Combin, 1998, 49: 280.

[10] Johnson R T, Vaughan T P. On union-closed families [J]. J Combin Theory Ser A, 1998, 84: 242.

[11] Lo Faro G. A note on the union-closed sets conjecture [J]. J Austral Math Soc Ser A, 1994, 57: 230

[12] Marković P. An attempt at Frankl's conjecture [J]. Publ I-Beograd Math, 2007, 81: 29.

[13] Morris R. FC-families and improved bound for Frankl's conjecture [J]. European J Combin, 2006, 27: 269.

[14] Vaughan T P. Families implying the Frankl conjecture [J]. European J Combin, 2002, 23: 851.

[15] Vaughan T P. A note on the union-closed sets conjecture [J]. J Combin Math Combin Comput, 2003, 45: 95.

[16] Vaughan T P. Three-sets in a union-closed family [J]. J Combin Math Combin Comput, 2004, 49: 73.

[17] Cui Z, Hu Z C. Two stronger versions of the union-closed sets conjecture [J]. arXiv, 2017, 1711.04276.



**引用本文格式:**

中　文：胡泽春，李世伦. $S_1$-Frankl 猜想的 6 元素情形（Ⅰ）[J]. 四川大学学报：自然科学版，2020，57：11.

英　文：Hu Z C, Li S L. The 6-element case of $S_1$-Frankl conjecture（Ⅰ）[J]. J Sichuan Univ: Nat Sci Ed, 2020, 57: 11.