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具有逐段常量的三阶微分方程解的周期性和概周期性

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摘　要：本文研究具有逐段常量的三阶微分方程

$$x'''(t)-a^2x'(t)=bx\left(2\left[\frac{t+1}{2}\right]\right),$$

通过方程对应的差分方程给出了方程解的具体形式，并由此得到了关于方程解的周期性和概周期性的一些结果.

关键词：三阶微分方程；逐段常量；周期性；概周期性
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Periodicity and almost periodicity for solutions of third-order differential equations with piecewise constant argument

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Abstract: In this paper, we consider the following third-order differential equation with piecewise constant argument:

$$x'''(t)-a^2x'(t)=bx\left(2\left[\frac{t+1}{2}\right]\right),$$

give the form of the solution in term of the solution of the corresponding difference equation. Then we give some results on the periodicity and almost periodicity for the solutions of the equation.

Keywords: Third-order differential equation; Piecewise constant argument; Periodicity; Almost periodicity
(2000 MSC 34K13; 34K14)

1　Introduction

Differential equations with piecewise constant arguments, introduced by Cooke, Wiener and Shah[1-2], have been studied intensively over the past few decades. Periodicity and almost periodicity of this kind of differential equations are attractive topics in the qualitative theory of differential equations due to their significance and applications in physicology, control theory and others. Therefore many works appeared in this field (see e.g. Refs. [3-10]).

In 1994, Papaschinopoulos and Schinas[11] studied the existence, uniqueness and asymptotic

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stability of solutions for the equation

$$(y(t)+py(t-1))'''=-qy\left(2\left[\frac{t+1}{2}\right]\right),$$

where $[\cdot]$ is the greatest integer function. In 2012, Zhuang[12] presented some results on the existence and uniqueness of almost periodic solutions of the following Nth-order neutral differential equations:

$$(x(t)+px(t-1))^{(N)}=qx([t])+f(t).$$

In 2013, Zhuang and Wu[13] studied the almost periodicity for the third-order neutral delay-differential equations of the form

$$(x(t)+px(t-1))'''=qx([t])+f(t).$$

Recently, Bereketoglu, Lafci and Oztepe[14] considered the oscillation, nonoscillation and periodicity of a third-order equation

$$x'''(t)-a^2x'(t)=bx([t-1]).$$

Motivated by the above-mentioned results, in this paper we consider the following system

$$\begin{cases} x'''(t)-a^2x'(t)=bx\left(2\left[\frac{t+1}{2}\right]\right), \\ x(-1)=\alpha_0, x'(-1)=\alpha_1, x''(-1)=\alpha_2 \end{cases} \quad (1)$$

with $a, b\in \mathbf{R}$ such that

$$(a^3+ab-ab\sinh a)(a^3-ab+b\sinh a)\neq 0 \quad (2)$$

2 Preliminaries

Let $\mathbf{Z}$, $\mathbf{N}$ and $\mathbf{R}$ denote the sets of all integers, positive integers and real numbers, respectively, and for $n\in\mathbf{N}$ and $p\in\mathbf{N}$, $\mathbf{Z}[n,n+p]=\mathbf{Z}\cap[n,n+p]$. We denote by $BC(\mathbf{R},\mathbf{R})$ the Banach space of all bounded continuous functions $f:\mathbf{R}\to\mathbf{R}$ with supremum norm and by $B(\mathbf{Z},\mathbf{R})$ the Banach space of all bounded sequences $\{c_n\}_n\in\mathbf{Z}$ with supremum norm. Now, we give some definitions and lemmas, which can be found(or simply deduced from the theory) in any book, say Ref. [15], on almost periodic functions.

Definition 2.1 A function $x:\mathbf{R}\to\mathbf{R}$ is said to be a solution of problem (1) if it satisfies the following conditions:

(i) x''' exists on $\mathbf{R}$ with the possible exception of the points $(2n-1)$, where the one-sided 3rd derivatives exist;

(ii) x satisfies (1) on each interval $[2n-1, 2n+1)$, $n\in\mathbf{Z}$.

Definition 2.2 A subset S of $\mathbf{R}$ is called relatively dense in $\mathbf{R}$ if there exists a positive number L such that $[a,a+L]\cap S\neq\emptyset$ for all $a\in\mathbf{R}$. A function $f\in BC(\mathbf{R},\mathbf{R})$ is said to be almost periodic if for every $\varepsilon>0$ the set

$$T(f,\varepsilon)=\{\tau: |f(t+\tau)-f(t)|<\varepsilon \text{ for all } t\in\mathbf{R}\}$$

is relatively dense in $\mathbf{R}$. We denote the set of such functions by $AP(\mathbf{R})$.

Definition 2.3 A set $P\in\mathbf{Z}$ is said to be relatively dence in $\mathbf{Z}$ if there exists a positive integer p such that $\mathbf{Z}[n,n+p]\cap P\neq\emptyset$ for all $n\in\mathbf{Z}$. A sequence $x\in B(\mathbf{Z},\mathbf{R})$ is said to be almost periodic if for every $\varepsilon>0$ the set

$$T(x,\varepsilon)=\{\tau\in\mathbf{Z}: |x(n+\tau)-x(n)| <\varepsilon \text{ for all } n\in\mathbf{Z}\}$$

is relatively dense in $\mathbf{Z}$. We denote the set of sequences by $APS(\mathbf{R})$.

Lemma 2.4 $x\in APS(\mathbf{R})$ if and only if there exists $f\in AP(\mathbf{R})$ such that $f(n)=x(n)$ for $n\in\mathbf{Z}$.

3 Main results

3.1 The form of the solution

Because of the piecewise constant arguments, by the Picard theorem for the classical ordinary differential equations, we can get easily the existence and uniqueness of the solution for system (1). Now we induce the difference equation corresponding to (1).

Let $x(t)$ be a solution of (1), and

$$x(n)=c_n, x'(n)=d_n, x''(n)=e_n, n\in\mathbf{Z}.$$

Then (1) reduces to

$$x'''(t)-a^2x'(t)=bc_{2n}, t\in[2n-1,2n+1), n\in\mathbf{Z}.$$

For $t\in[2n-1,2n+1)$, $n\in\mathbf{Z}$, it is well known that the solution of the above equation is given as

$$x(t)=K_n+L_n\cosh a(t-2n+1)+ M_n\sinh a(t-2n+1)-\frac{b}{a^2}tc_{2n} \quad (3)$$

with constants K_n, L_n and M_n. Letting $t=2n-1$ in (3), we have

$$c_{2n-1}=K_n+L_n-\frac{b}{a^2}(2n-1)c_{2n} \quad (4)$$

Taking $t=2n-1$ in the first and second derivatives of (3), respectively, we get

$$M_n=\frac{d_{2n-1}}{a}+\frac{b}{a^3}c_{2n},L_n=\frac{e_{2n-1}}{a^2} \tag{5}$$

From (4) and (5), we derive

$$K_n=c_{2n-1}-\frac{e_{2n-1}}{a^2}+\frac{b}{a^2}(2n-1)c_{2n} \tag{6}$$

Substituting (5) and (6) in (3), we have

$$x(t)=\frac{-1+\cosh a(t-2n+1)}{a^2}e_{2n-1}+\frac{\sinh a(t-2n+1)}{a}d_{2n-1}+c_{2n-1}$$
$$+\left(-\frac{b}{a^2}(t-2n+1)+\frac{b}{a^3}\sinh a(t-2n+1)\right)c_{2n} \tag{7}$$

and then

$$\begin{cases} x'(t)=\dfrac{\sinh a(t-2n+1)}{a}e_{2n-1}+\cosh a(t-2n+1)d_{2n-1}+\left(\dfrac{b}{a^2}\cosh a(t-2n+1)-\dfrac{b}{a^2}\right)c_{2n}, \\ x''(t)=\cosh a(t-2n+1)e_{2n-1}+a\sinh a(t-2n+1)d_{2n-1}+\dfrac{b}{a}\sinh a(t-2n+1)c_{2n} \end{cases} \tag{8}$$

By the continuity of $x(t)$, setting $t=2n+1$ in (7) and (8), it follows that

$$\begin{cases} c_{2n+1}=c_{2n-1}+\dfrac{\sinh 2a}{a}d_{2n-1}+\left(\dfrac{\cosh 2a}{a^2}-\dfrac{1}{a^2}\right)e_{2n-1}+\left(\dfrac{b\sinh 2a}{a^3}-\dfrac{2b}{a^2}\right)c_{2n}, \\ d_{2n+1}=(\cosh 2a)d_{2n-1}+\dfrac{\sinh 2a}{a}e_{2n-1}+\left(\dfrac{b\cosh 2a}{a^2}-\dfrac{b}{a^2}\right)c_{2n}, \\ e_{2n+1}=a(\sinh 2a)d_{2n-1}+(\cosh 2a)e_{2n-1}+\dfrac{b\sinh 2a}{a}c_{2n} \end{cases} \tag{9}$$

Meanwhile, setting $t=2n$ in (7) and (8), we get

$$\begin{cases} \left(1+\dfrac{b}{a^2}-\dfrac{b\sinh a}{a^3}\right)c_{2n}=c_{2n-1}+\dfrac{\sinh a}{a}d_{2n-1}+\left(\dfrac{\cosh a}{a^2}-\dfrac{1}{a^2}\right)e_{2n-1}, \\ \left(\dfrac{b}{a^2}-\dfrac{b\cosh a}{a^2}\right)c_{2n}+d_{2n}=(\cosh a)d_{2n-1}+\dfrac{\sinh a}{a}e_{2n-1}, \\ -\dfrac{b\sinh a}{a}c_{2n}+e_{2n}=a(\sinh a)d_{2n-1}+(\cosh a)e_{2n-1} \end{cases} \tag{10}$$

Let $\boldsymbol{v}_n=(c_n,d_n,e_n)^T$ and

$$\boldsymbol{A}=\begin{pmatrix} 1 & \dfrac{\sinh 2a}{a} & \dfrac{\cosh 2a}{a^2}-\dfrac{1}{a^2} \\ 0 & \cosh 2a & \dfrac{\sinh 2a}{a} \\ 0 & a(\sinh 2a) & \cosh 2a \end{pmatrix},$$

$$\boldsymbol{B}=\begin{pmatrix} \dfrac{b\sinh 2a}{a^3}-\dfrac{2b}{a^2} & 0 & 0 \\ \dfrac{b\cosh 2a}{a^2}-\dfrac{b}{a^2} & 0 & 0 \\ \dfrac{b\sinh 2a}{a} & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{C}=\begin{pmatrix} 1+\dfrac{b}{a^2}-\dfrac{b\sinh a}{a^3} & 0 & 0 \\ \dfrac{b}{a^2}-\dfrac{b\cosh a}{a^2} & 1 & 0 \\ -\dfrac{b\sinh a}{a} & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{D}=\begin{pmatrix} 1 & \dfrac{\sinh a}{a} & \dfrac{\cosh a}{a^2}-\dfrac{1}{a^2} \\ 0 & \cosh a & \dfrac{\sinh a}{a} \\ 0 & a\sinh a & \cosh a \end{pmatrix}.$$

Then (9) and (10) become

$$\boldsymbol{v}_{2n+1}=\boldsymbol{A}\,\mathrm{v}_{2n-1}+\boldsymbol{B}\,\boldsymbol{v}_{2n},\boldsymbol{C}\,\boldsymbol{v}_{2n}=\boldsymbol{D}\,\boldsymbol{v}_{2n-1}.$$

Let $\boldsymbol{w}_n=\boldsymbol{v}_{2n-1}$, we obtain the difference equation corresponding to (1):

$$\begin{cases} \boldsymbol{w}_{n+1}=(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{D})\boldsymbol{w}_n, n\in Z, \\ \boldsymbol{w}_0=(\alpha_0,\alpha_1,\alpha_2)^T \end{cases} \tag{11}$$

The matrix $\boldsymbol{C}$ is invertible because of condition (2). Then it is easy to get the characteristic equation of (11):

$$\lambda^3+\frac{-a^3-2a^3\cosh 2a+ab(1-2\cosh 2a)+b\sinh a}{a^3+ab-b\sinh a}\lambda^2+$$

$$\frac{a^3+2a^3\cosh 2a+ab(1-2\cosh 2a)+b\sinh a}{a^3+ab-b\sinh a}\lambda-\frac{a^3+ab+b\sinh a}{a^3+ab-b\sinh a}=0.$$

We notice that the eigenvalues $\lambda\neq 0$ by (2). Now we have the following theorem.

Theorem 3.1 The form of the solution of system (1) with condition (2) is as follows.

$$x(t)=(\boldsymbol{l}_1(t)+\boldsymbol{l}_2(t))\boldsymbol{w}_n,\quad t\in[2n-1,2n+1),n\in\mathbf{Z} \tag{12}$$

where

$$l_1(t)=\left(1,\frac{\sinh a(t-2n+1)}{a},\frac{-1+\cosh a(t-2n+1)}{a^2}\right),$$

$$l_2(t)=\left(-\frac{b}{a^2}(t-2n+1)+\frac{b}{a^3}\sinh a(t-2n+1),0,0\right)\boldsymbol{C}^{-1}\boldsymbol{D}$$

and $\boldsymbol{w}_n=(\boldsymbol{w}_0,\boldsymbol{P}\boldsymbol{w}_0,\boldsymbol{P}^2\boldsymbol{w}_0)\boldsymbol{Q}_n(\lambda)$ is the solution of (11). Here $\boldsymbol{P}=\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{D}$ and $\boldsymbol{Q}_n(\lambda)$ is a vector determined by the eigenvalues $\lambda_1,\lambda_2,\lambda_3$ in the following 3 cases:

(i) if all the eigenvalues are simple,

$$\boldsymbol{Q}_n(\lambda)=\begin{pmatrix}1&\lambda_1&\lambda_1^2\\1&\lambda_2&\lambda_2^2\\1&\lambda_3&\lambda_3^2\end{pmatrix}^{-1}\begin{pmatrix}\lambda_1^n\\\lambda_2^n\\\lambda_3^n\end{pmatrix} \tag{13}$$

(ii) if $\lambda_1=\lambda_2\neq\lambda_3$, then

$$\boldsymbol{Q}_n(\lambda)=\begin{pmatrix}1&\lambda_1&\lambda_1^2\\0&\lambda_1&2\lambda_1^2\\1&\lambda_3&\lambda_3^2\end{pmatrix}^{-1}\begin{pmatrix}\lambda_1^n\\n\lambda_1^n\\\lambda_3^n\end{pmatrix};$$

(iii) if $\lambda_1=\lambda_2=\lambda_3$, then

$$\boldsymbol{Q}_n(\lambda)=\begin{pmatrix}1&\lambda_1&\lambda_1^2\\0&\lambda_1&2\lambda_1^2\\0&\lambda_1&4\lambda_1^2\end{pmatrix}^{-1}\begin{pmatrix}\lambda_1^n\\n\lambda_1^n\\n^2\lambda_1^n\end{pmatrix}.$$

Proof It is obvious that (12) holds from (6) and (11). So we need only to prove that the solution w_n of (11) have the required form.

(i) If all the eigenvalues are simple, the solution of (11) can be given as

$$\boldsymbol{w}_n=\lambda_1^n\boldsymbol{k}_1+\lambda_2^n\boldsymbol{k}_2+\lambda_3^n\boldsymbol{k}_3 \tag{14}$$

where $\boldsymbol{k}_j=(k_{1j},k_{2j},k_{3j})^T$, $j=1,2,3$. Then we have

$$(\boldsymbol{w}_0,\boldsymbol{P}\boldsymbol{w}_0,\boldsymbol{P}^2\boldsymbol{w}_0)=(\boldsymbol{w}_0,\boldsymbol{w}_1,\boldsymbol{w}_2)=(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\begin{pmatrix}1&\lambda_1&\lambda_1^2\\1&\lambda_2&\lambda_2^2\\1&\lambda_3&\lambda_3^2\end{pmatrix},$$

which implies that

$$(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=(\boldsymbol{w}_0,\boldsymbol{P}\boldsymbol{w}_0,\boldsymbol{P}^2\boldsymbol{w}_0)\begin{pmatrix}1&\lambda_1&\lambda_1^2\\1&\lambda_2&\lambda_2^2\\1&\lambda_3&\lambda_3^2\end{pmatrix}^{-1}$$

and

$$\boldsymbol{w}_n=(\boldsymbol{w}_0,\boldsymbol{P}\boldsymbol{w}_0,\boldsymbol{P}^2\boldsymbol{w}_0)\begin{pmatrix}1&\lambda_1&\lambda_1^2\\1&\lambda_2&\lambda_2^2\\1&\lambda_3&\lambda_3^2\end{pmatrix}^{-1}\begin{pmatrix}\lambda_1^n\\\lambda_2^n\\\lambda_3^n\end{pmatrix}.$$

That is (13) holds.

(ii) If $\lambda_1=\lambda_2\neq\lambda_3$, the solution of (11) can be expressed by

$$\boldsymbol{w}_n=\lambda_1^n\boldsymbol{k}_1+n\lambda_1^n\boldsymbol{k}_2+\lambda_3^n\boldsymbol{k}_3,$$

where $\boldsymbol{k}_j=(k_{1j},k_{2j},k_{3j})^T$, $j=1,2,3$. Then (ii) can be proved by the same arguments of (i).

(iii) If $\lambda_1=\lambda_2=\lambda_3$, the solution of (11) can be written as

$$\boldsymbol{w}_n=\lambda_1^n\boldsymbol{k}_1+\mathrm{n}\lambda_1^{\mathrm{n}}\boldsymbol{k}_2+n^2\lambda_1^n\boldsymbol{k}_3,$$

where $\boldsymbol{k}_j=(k_{1j},k_{2j},k_{3j})^T$, $j=1,2,3$. Then, similarly to (i), we can prove (iii).

Lemma 3.2

From (11), one have

$$\begin{cases}d_{2n-1}=r_{11}c_{2n+3}+r_{12}c_{2n+1}+r_{13}c_{2n-1},\\ e_{2n-1}=r_{21}c_{2n+3}+r_{22}c_{2n+1}+r_{23}c_{2n-1}\end{cases} \tag{15}$$

Here the constants r_{ij}, $i=1,2$, $j=1,2,3$ only depend on a,b.

Proof According to (11), we have

$$c_{2n+3}=(1,0,0)(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{D})\begin{pmatrix}c_{2n+1}\\d_{2n+1}\\e_{2n+1}\end{pmatrix}.$$

This together with (11) forms a system of 4 equations with 4 variables $c_{2n-1},d_{2n-1},c_{2n+1},d_{2n+1}$. Then, by a fundamental calculation, we can get (15) with

$$r_{11}=-a/(2\sinh 2a),$$

$$r_{12}=a(-2a^3-2a^3\cosh 2a+b\sinh a-\frac{b\sinh 2a+b sinh 3a-2ab\cosh 2a)}{(2\sinh 2a(-a^3-ab+b\sinh a)),}$$

$r_{13} = a(a^3 + 2a^3\cosh 2a - ab + b\sinh 2a - \dfrac{b\sinh 3a + b\sinh 4a - 2ab\cosh 2a)}{(2\sinh 2a(-a^3 - ab + b\sinh a))}$,

$r_{21} = a^2/(4\sinh^2 a)$

$r_{22} = a^2(-2a^3\cosh 2a + 2ab - b\sinh a - \dfrac{b\sinh 2a + b\sinh 3a - 2ab\cosh 2a)}{(4\sinh^2 a(a^3 + ab - b\sinh a))}$,

$r_{23} = a^2(-a^3 + 2a^3\cosh 2a + ab + 2b\sinh a - \dfrac{b\sinh 2a - b\sinh 3a + b\sinh 4a - 2ab\cosh 2a)}{(4\sinh^2 a(a^3 + ab - b\sinh a))}$.

3.2 Periodicity and almost periodicity

We first consider the periodicity of the solution of problem (1).

Theorem 3.3 Let $k\in\mathbf{N}$, $x(t)$ be the solution of (1) and $\{\boldsymbol{w}_n\}_{n\in\mathbf{Z}} = \{c_{2n-1}, d_{2n-1}, e_{2n-1}\}^T_{n\in\mathbf{Z}}$ be the solution of (11). Then the following statements are true.

(i) $x(t)$ is $2k$-periodic if and only if $\{\boldsymbol{w}_n\}_{n\in\mathbf{Z}}$ is k-periodic;

(ii) $x(t)$ is $2k$-periodic if and only if $\{c_{2n-1}\}_{n\in\mathbf{Z}}$ is k-periodic.

Proof (i) If $x(t+2k) = x(t)$ for $t\in\mathbf{R}$, it is easy to get from the definition of the solution $\{\boldsymbol{w}_n\}_{n\in\mathbf{Z}}$ of (12) that $\{\boldsymbol{w}_n\}_{n\in\mathbf{Z}}$ is k-periodic. Conversely, by Theorem 3.1, it is easy to see that $\boldsymbol{l}_1(t)$ and $\boldsymbol{l}_2(t)$ are 2-periodic. Suppose that $\{w_n\}_{n\in\mathbf{Z}}$ is k-periodic. Then it follows from (12) that $x(t)$ is $2k$-periodic.

(ii) If $x(t)$ is $2k$-periodic, by (i), we can see that $\{c_{2n-1}\}_{n\in\mathbf{Z}}$ is k-periodic. Conversely, suppose that $\{c_{2n-1}\}_{n\in\mathbf{Z}}$ is k-periodic. Then $\{d_{2n-1}\}_{n\in\mathbf{Z}}$ and $\{e_{2n-1}\}_{n\in\mathbf{Z}}$ are k-periodic by (15). Thus $\{\boldsymbol{w}_n\}_{n\in\mathbf{Z}}$ is k-periodic. So $x(t)$ is $2k$-periodic by (i).

For the almost periodicity of the solution of problem (1), we have the following result.

Theorem 3.4 Let $k\in\mathbf{N}$, $x(t)$ be the solution of (1) and $\{\boldsymbol{w}_n\}_{n\in\mathbf{Z}} = \{c_{2n-1}, d_{2n-1}, e_{2n-1}\}^T_{n\in\mathbf{Z}}$ be the solution of (11). Then the following statements hold:

(i) $x\in AP(\mathbf{R})$ if $\{\boldsymbol{w}_n\}_{n\in\mathbf{Z}}\in \mathrm{VAPS}(\mathbf{R})$;

(ii) $x\in AP(\mathbf{R})$ if and only if $\{c_{2n-1}\}_{n\in\mathbf{Z}}\in APS(\mathbf{R})$.

Proof (i) Assume that $\{\boldsymbol{w}_n\}_{n\in\mathbf{Z}}\in APS(\mathbf{R})$. By Theorem 3.1, it is easy to see that $\boldsymbol{l}_1(t)$ and $\boldsymbol{l}_2(t)$ are periodic. Thus $\boldsymbol{l}_1, \boldsymbol{l}_2\in APS(\mathbf{R})$, and then we get $x\in AP(\mathbf{R})$ from (12).

(ii) If $x\in AP(\mathbf{R})$, it is easy to see that $x(2t-1)$ is also almost periodic in t. Then $\{c_{2n-1}\}_{n\in\mathbf{Z}} = \{x(2n-1)\}_{n\in\mathbf{Z}}\in APS(\mathbf{R})$. Conversely, if $\{c_{2n-1}\}_{n\in\mathbf{Z}}\in APS(\mathbf{R})$, we have $\{d_{2n-1}\}_{n\in\mathbf{Z}}$, $\{e_{2n-1}\}_{n\in\mathbf{Z}}\in APS(\mathbf{R})$ by (15), and then $\{w_n\}_{n\in\mathbf{Z}}\in APS(\mathbf{R})$. By (i), we have $x\in AP(\mathbf{R})$.

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