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具平坦欧氏边界的局部凸浸入超曲面

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摘　要：在仿射微分几何中，局部强（一致）凸的浸入超曲面的几何与拓扑性质非常复杂．本文构造出一类新的局部强凸的浸入超曲面，其欧氏边界是平坦的（即边界落在一个超平面内），但曲面本身却不是整体凸的．这与目前已有的结论完全不同．

关键词：浸入超曲面，欧氏边界点，局部强凸
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Locally convex immersed surface with flat Euclidean boundary

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Abstract: In differential geometry, the geometric and topological behaviors of locally strongly (uniformly) convex immersed surfaces (hypersurfaces) are very complicated, so are their Euclidean boundaries. In this paper, we construct new locally strongly convex (but not globally convex) immersed surfaces (hypersurfaces) with flat Euclidean boundary in $\mathbf{R}^{n+1}$ ($n=2,3$), which are different from an existing conclusion.

Keywords: Immersed surface; Euclidean boundary point; Locally strongly convex
(2010 MSC 53A15; 35J60; 35J65; 53C42)

1 Introduction

Firstly, we recall some notions on an immersed hypersurface in differential geometry. An immersed hypersurface $\mathscr{M}$ is defined as $x: \boldsymbol{M} \to \mathbf{R}^{n+1}$, where $\boldsymbol{M}$ is an n-dimensional differential manifold.

(i) If for each $p \in \boldsymbol{M}$, there is a neighborhood $\mathscr{U}_p \subset \boldsymbol{M}$ such that $x(\mathscr{U}_p)$ lies on one side of the tangent hyperplane π at $x(p)$, then we call $x(M)$ a locally convex hypersurface.

(ii) For a locally convex surface $x(M)$, if $\pi \cap x(\mathscr{U}_p) = \{x(p)\}$ for each $p \in M$, then we call $x(M)$ the locally strongly (strictly) convex hypersurface.

If $x(M)$ is C^2, the definition “locally strongly convex” is equivalent to “locally uniformly convex”. Generally, to prove that $x(M)$ is locally strongly (uniformly) convex hypersurface, we only needs to prove that its second fundamental form is positive definite.

(iii) If for each $p \in M$, $x(M)$ lies on one side of its tangent hyperplane at $x(p)$, then we call $x(M)$ a globally convex hypersurface, or (simply) a convex hypersurface.

(iv) For a locally convex immersed hypersurface $x(M)$, we call a point $P \in \mathbf{R}^{n+1}$ the Euclidean boundary point of $x(M)$, if $P \in \overline{x(M)} \setminus x(M)$,

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where $\overline{x(M)}$ is the closure of $x(M)$ with respect to its Euclidean topology. We will denote by $\partial\mathscr{M}=\overline{x(M)}\backslash x(M)$ the set of the Euclidean boundary point of $x(M)$.

An equivalent definition of the immersed locally convex hypersurface is stated in Ref. [1].

Definition 1.1 A locally convex surface $\mathscr{M}$ in $\mathbf{R}^{n+1}$ is an immersion of n-dimensional oriented and connected manifold $\mathscr{N}$ (possibly with boundary) in $\mathbf{R}^{n+1}$, i. e., a mapping $T:\mathscr{N}\to\mathscr{M}\subset\mathbf{R}^{n+1}$, such that for any $p\in\mathscr{N}$, there exists a neighborhood $\omega_p\subset\mathscr{N}$ such that

(i) T is a homeomorphism from ω_p to $T(\omega_p)$;

(ii) $T(\omega_p)$ is a convex graph;

(iii) the convexity of $T(\omega_p)$ agrees with the orientation.

A hypersurface is assumed to be locally uniformly convex, namely it has positive principal curvatures[2].

Generally the geometric property of a hypersuface boundary is closely related to the completeness of the immersed hypersuface itself. So it is interesting to study the boundary character of a locally uniformly (strongly) convex immersed hypersurface. In Ref. [3], the author classifies the Euclidean boundary points as two classes and gives many hypersurfaces with the first class and the second class Euclidean boundary point respectively.

In this paper, we will prove that there exist locally strongly convex immersed surfaces in $\mathbf{R}^3$ such that their boundaries lie in a plane but the surfaces is not globally convex, which are different from an existing conclusion. Moreover, we give a method to construct similar hypersurfaces in $\mathbf{R}^4$, i. e., we get the following theorem:

Theorem 1.2 There is a locally convex hypersurface $\mathscr{M}$ in $\mathbf{R}^{n+1}, n\geqslant 2$ such that $\mathscr{M}$ is strictly convex at some point p_0 and the boundary $\partial\mathscr{M}$ lies on a hyperplane $\hat{P}$ but $\mathscr{M}$ is not convex (globally).

For $n=2$, there is an Example 1.3 constructed by author and involved in Ref. [4], it is totally different from Lemma 2.1 of Refs. [5-6].

Example 1.3 Let $M=\{(u,t)\mid(u,t)\in(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbf{R}\}\subset\mathbf{R}^2$, $x:M\to\mathbf{R}^3$ be an immersed surface in $\mathbf{R}^3$ defined by

$$\vec{r}(u,t)=(\mathrm{e}^{-\frac{t^2}{a}}\cos t\cos u,\mathrm{e}^{-\frac{t^2}{a}}\sin t\cos u,\sin u) \quad (1)$$

where $a>0$ be a constant. The Euclidean boundary of $x(M)$ is a line segment

$$\partial\mathscr{M}=\left\{(0,0,\sin u)\in\mathbf{R}^3\mid u\in[-\frac{\pi}{2},\frac{\pi}{2}]\right\}.$$

The surface is convex at all the points corresponding to $t=0$, i. e., at any points of the half circle

$$\{(\cos u,0,\sin u)\mid-\frac{\pi}{2}<u<\frac{\pi}{2}\}\subset x(M).$$

$x(M)$ is obviously not globally convex (see Fig. 1 and Fig. 2).

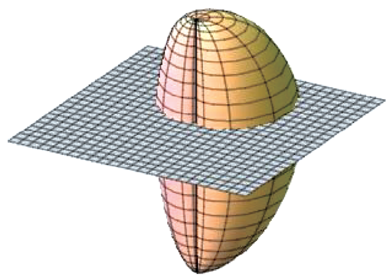

Fig. 1 A sketch map of $x(M)$ for $a=2$

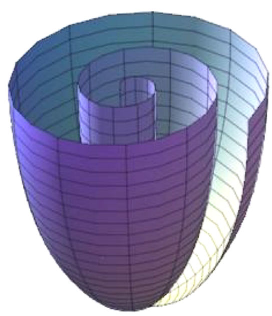

Fig. 2 The cross-section drawn of $x(M)$ for $a=100$, $-5\pi<t<0$, $-\frac{\pi}{2}<u<0$

In the next section, we will construct a new locally strongly (uniformly) convex surface in $\mathbf{R}^3$ ($n=2$) and some new examples for $n>2$ which are satisfying Theorem 1.2. Then we complete the proof.

2 New examples

Let $M\subset\mathbf{R}^2$ be an open set and $x:M\to\mathbf{R}^3$ be an immersed surface in $\mathbf{R}^3$ defined by

$$\vec{r}(u,t)=(f(u)g(t)\cos t,$$

$f(u)g(t)\sin t, h(u))$ (2)

where $f(u), g(t), h(u)$ are smooth functions to be determined later. By direct calculations we have

$\vec{r}_u = (f'g\cos t, f'g\sin t, h')$,

$\vec{r}_t = f(g'\cos t - g\sin t, g'\sin t + g\cos t, 0)$.

A normal vector of $x: M \to \mathbf{R}^3$ is

$\vec{r}_u \times \vec{r}_t = (-h'f(g'\sin t + g\cos t),$
$h'f(g'\cos t - g\sin t), ff'g^2)$.

Denote

$|\vec{r}_u \times \vec{r}_t| = |f|[h'^2(g'^2+g^2)+f'^2g^4]^{\frac{1}{2}} :=$
$|f|A$,

where $A := [h'^2(g'^2+g^2)+f'^2g^4]^{\frac{1}{2}}$.

Next, we have

$\vec{r}_{uu} = (f''g\cos t, f''g\sin t, h'')$,

$\vec{r}_{ut} = f'(g'\cos t - g\sin t, g'\sin t + g\cos t, 0)$,

$\vec{r}_{tt} = f(g''\cos t - 2g'\sin t - g\cos t, g''\sin t +$
$2g'\cos t - g\sin t, 0)$.

Then we get the second fundamental form of $x: M \to \mathbf{R}^3$, which can be expressed as

$II = L_{uu}\,\mathrm{d}u^2 + 2L_{ut}\,\mathrm{d}u\mathrm{d}t + L_{tt}\,\mathrm{d}t^2$,

where

$L_{uu} = A^{-1}g^2(h''f' - h'f'')$, $L_{tt} =$
$A^{-1}fh'(-gg'' + 2g'^2 + g^2)$, $L_{ut} = 0$ (3)

Define the function $g(t)$ as following.

$$g(t) = \begin{cases} 1, & \text{if } t \leqslant -\frac{\pi}{2}, \\ 1 + \frac{1}{b}\left(t^2 - \frac{\pi^2}{4}\right)^4, & \text{if } -\frac{\pi}{2} < t < \frac{\pi}{2}, \\ 1, & \text{if } t \geqslant \frac{\pi}{2} \end{cases} \quad (4)$$

where $b > 0$ is a constant to be determined later. A direct calculation shows that $g(t)$ is C^2. For $-\frac{\pi}{2} < t < \frac{\pi}{2}$, we have

$$-gg'' + 2g'^2 + g^2 = g^2 + 2g'^2 - g\left\{\frac{8}{b}\left(t^2 - \frac{\pi^2}{4}\right)^3 + \frac{48t^2}{b}\left(t^2 - \frac{\pi^2}{4}\right)^2\right\} \geqslant g\left(1 - \frac{3\pi^4}{16b}\right) \quad (5)$$

Taking $b \geqslant \frac{3\pi^4}{8}$, we get $-gg'' + 2g'^2 + g^2 > 0$ holds for any $t \in (-\infty, +\infty)$.

Let $f(u) = u$, $h(u) = \mathrm{e}^u$, $u > 0$ in (2). We get

$L_{uu} = A^{-1}g^2\mathrm{e}^u > 0$ (6)

$L_{tt} = A^{-1}u\,\mathrm{e}^u(-gg'' + 2g'^2 + g^2) > 0$ (7)

let $M = \{(u,t) \mid (u,t) \in (0, +\infty) \times \mathbf{R}\} \subset \mathbf{R}^2$, $x: M \to \mathbf{R}^3$ be defined by

$\vec{r}(u,t) = (ug(t)\cos t, ug(t)\sin t, \mathrm{e}^u)$ (8)

for the surface $x(M) \subset \mathbf{R}^3$.

- By (6), (7), it is locally strongly convex.
- When $u \to 0$, $(0,0,1)$ is the unique Euclidean boundary point of the surface (see Fig. 3), so the boundary of $x(M)$ lies in a plane.

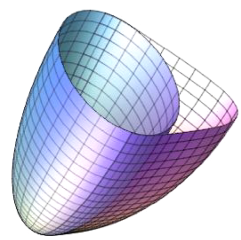

Fig. 3 The unique Euclidean boundary point of the surface

- The surface has many self-intersecting points at $t = \pm\frac{\pi}{2}$.
- The surface is convex at all the points corresponding to $-\frac{\pi}{2} < t < \frac{\pi}{2}$, but the surface is not globally convex.

Therefore, This example satisfies Theorem 1.2.

It is easy to construct the high dimensional hypersufaces satisfying Theorem 1.2. Here we give a method as follows (here we only consider $n=3$).

Let $M = \{(u,t,v) \mid (u,t,v) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbf{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})\} \subset \mathbf{R}^3$, the immersion $x: M \to \mathbf{R}^4$ is defined by

$\vec{r}(u,t,v) = (p(v)f(u)g(t)\cos t,$
$p(v)f(u)g(t)\sin t, h(u), k(v))$ (9)

as before, where

$f(u) = \cos u$, $h(u) = \sin u$, $g(t) = \mathrm{e}^{-\frac{t^2}{a}}$,

and $p(v), k(v)$ will be determined later. We have

$\vec{r}_u = (pf'g\cos t, pf'g\sin t, h', 0)$,

$\vec{r}_t = pf(g'\cos t - g\sin t, g'\sin t + g\cos t, 0, 0)$,

$\vec{r}_v = (p'fg\cos t, p'fg\sin t, 0, k')$.

The normal vector of $x: M \to \mathbf{R}^4$ is

$\vec{r}_u \wedge \vec{r}_t \wedge \vec{r}_v = (-k'h'pf(g'\sin t + g\cos t),$
$k'h'pf(g'\cos t - g\sin t), k'p^2ff'g^2,$
$pp'f^2g^2h')$.

Denote $\vec{n}=\vec{r}_u\wedge\vec{r}_t\wedge\vec{r}_v$,

$$|\vec{n}|=|fp|[k'^2h'^2(g'^2+g^2)+(k'^2p^2f'^2+p'^2f^2h'^2)g^4]^{\frac{1}{2}}:=|fp|A,$$

where

$$A:=[k'^2h'^2(g'^2+g^2)+(k'^2p^2f'^2+p'^2f^2h'^2)g^4]^{\frac{1}{2}}.$$

We have

$$\vec{r}_{uu}=(pf''g\cos t,pf''g\sin t,h'',0),$$

$$\vec{r}_{tt}=pf(g''\cos t-2g'\sin t-g\cos t,g''\sin t+2g'\cos t-g\sin t,0,0),$$

$$\vec{r}_{vv}=(p''fg\cos t,p''fg\sin t,0,k''),$$

$$\vec{r}_{ut}=pf'(g'\cos t-g\sin t,g'\sin t+g\cos t,0,0),$$

$$\vec{r}_{uv}=p'f'(g\cos t,g\sin t,0,0),$$

$$\vec{r}_{tv}=p'f(g'\cos t-g\sin t,g'\sin t+g\cos t,0,0).$$

Then we get the coefficients of the second fundamental forms of $x:M\to\mathbf{R}^4$ as follows

$$L_{uu}=\frac{\vec{r}_{uu}\cdot\vec{n}}{|\vec{n}|}=A^{-1}k'pg^2(h''f'-h'f'') \quad (10)$$

$$L_{tt}=\frac{\vec{r}_{tt}\cdot\vec{n}}{|\vec{n}|}=A^{-1}k'pfh'(-gg''+2g'^2+g^2) \quad (11)$$

$$L_{vv}=\frac{\vec{r}_{vv}\cdot\vec{n}}{|\vec{n}|}=A^{-1}fg^2h'(k''p'-k'p'') \quad (12)$$

$$L_{uv}=\frac{\vec{r}_{uv}\cdot\vec{n}}{|\vec{n}|}=-A^{-1}p'f'k'h'g^2 \quad (13)$$

$$L_{ut}=L_{vt}=0 \quad (14)$$

For $f(u)=\cos u,h(u)=\sin u,g(t)=e^{-\frac{t^2}{a}}$, by taking $p(v)=\cos v,k(v)=\sin v$, we get

$$L_{uu}>0,L_{tt}>0,L_{vv}>0 \quad (15)$$

$$L_{uu}L_{vv}-L_{uv}^2=A^{-2}g^4\cos^2u\cos^2v(1-\sin^2u\sin^2v)>0 \quad (16)$$

where

$$-gg''+2g'^2+g^2=e^{-\frac{2t^2}{a}}\left(1+\frac{2}{a}+\frac{4t^2}{a^2}\right)>0,$$

$a=\text{constant}>0$ and $u,v\in(-\frac{\pi}{2},\frac{\pi}{2})$. By (15), (16), we know that the following matrix is positive definite:

$$\begin{pmatrix}L_{uu} & 0 & L_{uv}\\ 0 & L_{tt} & 0\\ L_{uv} & 0 & L_{vv}\end{pmatrix} \quad (17)$$

This means that the immersion $x:M\to\mathbf{R}^4$ is locally strongly convex. The boundary $\partial x(M)$ is

$$\partial x(M)=\{(0,0,\sin u,\sin v)\in\mathbf{R}^4\,|\,u,v\in[-\frac{\pi}{2},\frac{\pi}{2}]\},$$

lies in a plane. One may easily check that the hypersuface defined by the above formula (9) satisfies Theorem 1.2.

Similarly, based on the above new example, one may get

$$\vec{r}(u,t,v)=(u\cos v g(t)\cos t,u\cos v g(t)\sin t,e^u,\sin v) \quad (18)$$

where $g(t)$ is defined by (4), $u>0,v\in(-\frac{\pi}{2},\frac{\pi}{2})$. It is also a locally strongly convex immersed hypersurfaces in $\mathbf{R}^4$ with flat Euclidean boundary and satisfies Theorem 1.2.

References:

[1] Trudinger N S, Wang X J. On locally convex hypersurfaces with boundary [J]. J Reine Angew Math, 2002: 551: 11.

[2] Trudinger N S, Wang X J. Affine complete locally convex hypersurfaces [J]. Invent Math, 2002, 150: 45.

[3] Wang B F. Some remarks on Euclidean boundary points of locally convex immersed hypersurfaces [J]. J Sichuan Univ: Nat Sci Ed, 2014, 51: 16.

[4] Li A M, Simon U, Zhao G, *et al*. Global affine differential geometry of hypersurfaces, second revised and extended edition [M]. Berlin: De Gruyter, 2015.

[5] Trudinger N, S, Wang X J. The Monge-Ampère equation and its geometric application, handbook of geometric analysis [M]. Beijing: Higher Education Press, 2008.

[6] Wang X J. Affine maximal hypersurfaces [M]. Beijing: Higher Education Press, 2002.

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