非线性 Schrödinger 方程长时间稳定性的一个估计

李海阔
(四川大学数学学院，成都 610064)

摘　要：本文考虑一维环面上的非线性 Schrödinger 方程 $\mathrm{i}\dot{\psi}=-\Delta\psi+F(|\psi|^2)\psi, x\in\mathbf{T}$ 的稳定性，其中 $\mathbf{T}:=\mathbf{R}/2\pi\mathbf{Z}$，$F:\mathbf{R}\to\mathbf{R}$ 解析，$F(0)=0$，$F'(0)\neq 0$. 在正则性指标 s 满足 $\frac{1}{2}<s<1$ 的条件下，本文给出了方程的小振幅解关于尺度为 ε 的初值($\varepsilon>0$ 且足够小)的长时间稳定性的一个估计. 具体而言，本文证明了方程解的傅里叶系数的模在 $\varepsilon^{-\left(4+\frac{2}{9}s\right)}$ 尺度的时间内几乎保持不变.

关键词：非线性 Schrödinger 方程；小振幅解；长时间稳定性
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An estimation of the long time stability of the nonlinear Schrödinger equation

LI Hai-Kuo
(School of Mathematics, Sichuan University, Chengdu 610064, China)

Abstract: In this paper, the stability of the nonlinear Schrödinger equation $\mathrm{i}\dot{\psi}=-\Delta\psi+F(|\psi|^2)\psi, x\in\mathbf{T}$ is considered on the one-dimensional torus, where $\mathbf{T}:=\mathbf{R}/2\pi\mathbf{Z}$, $F:\mathbf{R}\to\mathbf{R}$ is analytic, $F(0)=0$ and $F'(0)\neq 0$. Under the regularity index s fulfilling that $\frac{1}{2}<s<1$, we give an estimate for the long time stability of small amplitude solutions with initial data of size ε, where ε is positive and sufficiently small. More precisely, we prove that the modulus of the Fourier coefficients of solutions is approximately constant for the time order $\varepsilon^{-\left(4+\frac{2}{9}s\right)}$.

Keywords: Nonlinear Schrödinger equation; Small amplitude solutions; Long time stability
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1　Introduction

We consider the Nonlinear Schrödinger equation (NLS equation in brief) on the one-dimensional torus

$$\mathrm{i}\dot{\psi}=-\Delta\psi+F(|\psi|^2)\psi,\ x\in\mathbf{T}:=\mathbf{R}/2\pi\mathbf{Z} \tag{1}$$

where $F:\mathbf{R}\to\mathbf{R}$ is an analytic function on a neighborhood of the origin, $F(0)=0$ and $F'(0)\neq 0$, *i.e.* the nonlinear part of (1) is a cubic perturbation. In high regularity, it is well known that we can construct a canonical transformation which transforms the Hamiltonian functions of the partial differential equations into corresponding Birkhoff normal forms. Birkhoff normal form for long time dynamic behavior of Hamiltonian partial differential equations has been widely investigated by many authors, seeing Refs. [1-11] for example. In these examples, the NLS equation is stud-

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作者简介：李海阔(1997－)，男，四川成都人，硕士研究生，主要研究方向为哈密顿动力系统. E-mail：lhkfromswpu@163.com

ied by using the internal parameters in Refs. [4, 7]. Specifically, rational normal form is established for the NLS equation in Ref. [4]. Additionally in Ref. [7], normal forms are established by taking the initial value as parameters.

However, there are few results in low regularity. The canonical transformation concerning Hamiltonian functions is unavailable under $s<1$. Consequently, long time stability for solutions of equations cannot be obtained by previous technique. In Ref. [12], Bambusi introduces a method of constructing approximate integrals of motion concerning Hamiltonian functions and proves the long time stability of solutions of a defocusing NLS equation with $0<s<\frac{1}{2}$. Compared with our conclusion (Theorem 1.1), the results in Ref. [12] is a large probability averaging theorem, which considers lower regularity and exploits the invariance of Gibbs measure with respect to the defocusing equation. According to Refs. [13, 14], the invariant Gibbs measure could be established only when $s<\frac{1}{2}$. In our conclusion, we consider $\frac{1}{2}<s<1$, which indicates that the invariant Gibbs measure couldn't be set up. And the nonlinear Schrödinger equation whose form is more general. Besides, we obtain the longer stability time as well.

Consider the Fourier form of $\psi\in H^s$,

$$\psi=\sum_{k\in\mathbf{Z}}\psi_k(t)\mathrm{e}^{\mathrm{i}kx},\ \psi_k=\frac{1}{2\pi}\int_0^{2\pi}\psi(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x.$$

The Sobolev norm of ψ is $\|\psi\|_s^2=\sum_{k\in\mathbf{Z}}\langle k\rangle^{2s}|\psi_k|^2$, where $\langle k\rangle=\sqrt{1+k^2}$.

In the following theorem, we study the long time stability of small amplitude solutions with the low regularity for equation (1).

Theorem 1.1 For $\frac{1}{2}<s<1$ and $k\in\mathbf{Z}$, there exist constants C, $\varepsilon^*>0$ such that if the initial datum $\psi(0)\in H^s$ fulfills

$$\varepsilon:=\|\psi(0)\|_s<\varepsilon^* \tag{2}$$

then the solution ψ of equation (1) has a large probability of satisfying

$$\langle k\rangle^{2s}\left||\psi_k(t)|^2-|\psi_k(0)|^2\right|\leqslant\varepsilon^2 \tag{3}$$

for $t\leqslant C\varepsilon^{-(4+\frac{2}{9}s)}$.

Remark 1 For equation (1), we note that $F'(0)\neq 0$ in advance, since we need to eliminate more terms by using four-order normal form in the subsequent proof process. Meanwhile, the proof of Theorem 1.1 is based on the construction of approximate integrals of motion in Ref. [12]. For convenience, we keep up with some notations and terminology of construction process from Ref. [12].

The paper is organized as follows. In Section 2, we substitute $\varepsilon\psi$ for ψ to carry out scale transformation in order to simplify the subsequent proof process. Given $k\in\mathbf{Z}$, we construct the approximate integral of motion with regard to the Hamiltonian function of equation (1). For elimination of four and six order non-resonant parts, the form is consistent with the construction in Ref. [12]. Afterwards, we take the initial value as parameters to eliminate six-order terms which are resonant and informal. The idea is introduced by Bourgain in Ref. [7]. At this point, for resonant indexes $(k_1,k_2,k_3,k_4,k_5,k_6)\in\mathbf{Z}^6$, the small divisors $\xi_{k_1}+\xi_{k_2}+\xi_{k_3}-\xi_{k_4}-\xi_{k_5}-\xi_{k_6}$ appear here. In order to obtain corresponding results, it's necessary to impose a Diophantine condition. As a consequence, we can obtain the approximate integral of motion. And then we complete the estimates of the remainder terms. In Section 3, we complete the measure estimate concerning parameters of the small denominators. Therefore, we define the Gaussian measure with $\frac{1}{2}<s<1$. Finally, combining the remainder estimates and the result of the measure estimate, Theorem 1.1 is completely proved in Section 4.

2 Construction of the approximate integral of motion

Firstly, we substitute $\varepsilon\psi$ for ψ. After scaling, equation (1) can be rewritten as

$$\mathrm{i}\dot{\psi}=-\Delta\psi+\varepsilon^2F'(0)|\psi|^2\psi+\frac{\varepsilon^4F''(0)}{2!}|\psi|^4\psi+$$

$h.o.t$

where $h.o.t$ denotes higher terms. The Hamiltonian of equation (1) is written as

$$H = H_2 + \sum_{j=2}^{\infty} H_{2j} \tag{4}$$

where

$$H_2 = \frac{1}{2\pi}\int_0^{2\pi} |\nabla \psi(x)|^2 \mathrm{d}x,$$
$$H_4 = \frac{F'(0)}{2!}\frac{\varepsilon^2}{2\pi}\int_0^{2\pi}|\psi(x)|^4 \mathrm{d}x \tag{5}$$

$$H_{2j} = \frac{F^{(j-1)}(0)}{j!}\frac{\varepsilon^{2j-2}}{2\pi}\int_0^{2\pi}|\psi(x)|^{2j}\mathrm{d}x \tag{6}$$

Define the Hilbert space $l_s^2(\mathbf{Z},\mathbf{C})$ of the complex-valued sequence $z=(\psi_k)_{k\in\mathbf{Z}}$, with

$$\|z\|_s^2 = \sum_{k\in\mathbf{Z}}\langle k\rangle^{2s}|\psi_k|^2 < \infty.$$

The Sobolev norm $\|\psi\|_s$ is equivalent to $\|z\|_s$. Correspondingly, one has

$$H_2 = \sum_{k\in\mathbf{Z}} k^2 |\psi_k|^2,$$
$$H_4 = \frac{\varepsilon^2 F'(0)}{2}\sum_{\substack{(k_1,k_2,k_3,k_4)\in\mathbf{Z}^4\\ k_1+k_2=k_3+k_4}} \psi_{k_1}\psi_{k_2}\bar{\psi}_{k_3}\bar{\psi}_{k_4} \tag{7}$$

and

$$H_{2j} = \frac{\varepsilon^{2j-2}F^{(j-1)}(0)}{j!}\sum\prod_{q=1}^{j}\psi_{k_q}\prod_{q=j+1}^{2j}\bar{\psi}_{k_q},\ j\geqslant 2 \tag{8}$$

Define

$$\xi_k = |\psi_k(0)|^2 \tag{9}$$
$$I_k = |\psi_k|^2 \tag{10}$$
$$J_k = I_k - \xi_k \tag{11}$$

The Possion bracket of two Hamiltonian functions F and G is

$$\{F,G\} = -\mathrm{i}\sum_{k\in\mathbf{Z}}\left(\frac{\partial F}{\partial\psi_k}\frac{\partial G}{\partial\bar{\psi}_k} - \frac{\partial F}{\partial\bar{\psi}_k}\frac{\partial G}{\partial\psi_k}\right).$$

For any $n\in\mathbf{N}_+$, we denote the resonant index set by

$$M_{2n} = \{\boldsymbol{k}=(k_1,\cdots,k_{2n})\in\mathbf{Z}^{2n} \mid \sum_{j=1}^{n}k_j = \sum_{j=n+1}^{2n}k_j, \sum_{j=1}^{n}k_j^2 = \sum_{j=n+1}^{2n}k_j^2\}.$$

Define the operator $L_{H_2}=\{H_2,\cdot\}$, and L_{H_2} is an invertible operator. For a given polynomial F, $F^{N_{H_2}}$ and $F^{R_{H_2}}$ represent resonant and non-resonant parts of F, respectively. For instance,

$$H_4^{N_{H_2}} = Z_4 = \frac{\varepsilon^2 F'(0)}{2}\sum_{\boldsymbol{k}\in M_4}\psi_{k_1}\psi_{k_2}\bar{\psi}_{k_3}\bar{\psi}_{k_4},$$

$$H_4^{R_{H_2}} = \frac{\varepsilon^2 F'(0)}{2}\sum_{\substack{k_1+k_2=k_3+k_4\\ k_1^2+k_2^2\neq k_3^2+k_4^2}}\psi_{k_1}\psi_{k_2}\bar{\psi}_{k_3}\bar{\psi}_{k_4}.$$

For a given $\mathrm{k}\in\mathbf{Z}$, define

$$\Phi_{\mathrm{k},2} = |\psi_{\mathrm{k}}|^2 \tag{12}$$

and

$$\chi_4 = -L_{H_2}^{-1}H_4^{R_{H_2}},$$
$$\chi_6 = -L_{H_2}^{-1}\left(\frac{1}{2}\{\chi_4,H_4^{R_{H_2}}\}+\{\chi_4,Z_4\}+H_6\right)^{R_{H_2}} \tag{13}$$

$$Z_6 = H_6^{N_{H_2}} + \left(\frac{1}{2}\{\chi_4,H_4^{R_{H_2}}\}+\{\chi_4,Z_4\}\right)^{N_{H_2}} \tag{14}$$

According to (12)~(14), we define

$$\Phi_{\mathrm{k},4} = L_{\chi_4}|\psi_{\mathrm{k}}|^2,\ \Phi_{\mathrm{k},6} = \frac{1}{2}L_{\chi_4}^2|\psi_{\mathrm{k}}|^2 + L_{\chi_6}|\psi_{\mathrm{k}}|^2 \tag{15}$$

Write

$$Z_6 = \varepsilon^4\sum_{\boldsymbol{k}\in M_6}\tilde{Z}_{6,\boldsymbol{k}}\psi_{k_1}\psi_{k_2}\psi_{k_3}\bar{\psi}_{k_4}\bar{\psi}_{k_5}\bar{\psi}_{k_6}.$$

Considering that

$$R_6 = \{Z_6,\Phi_{\mathrm{k},2}\} \tag{16}$$

we get

$$R_6 = -\mathrm{i}\varepsilon^4\sum_{\boldsymbol{k}\in M_6}\tilde{Z}_{6,\boldsymbol{k}}\left(\sum_{j=1}^{3}\delta_{k_j,\mathrm{k}} - \sum_{j=4}^{6}\delta_{k_j,\mathrm{k}}\right)\psi_{k_1}\psi_{k_2}\psi_{k_3}\bar{\psi}_{k_4}\bar{\psi}_{k_5}\bar{\psi}_{k_6} \tag{17}$$

where $\delta_{k_j,\mathrm{k}} = \begin{cases}1,\ j=\mathrm{k},\\ 0,\ j\neq\mathrm{k}.\end{cases}$ The above process can be found in Ref. [12].

Z_4 can be computed directly and is given by

$$Z_4 = \varepsilon^2 F'(0)\left[\left(\sum_{k\in\mathbf{Z}}I_k\right)^2 - \frac{1}{2}\sum_{k\in\mathbf{Z}}I_k^2\right] \tag{18}$$

Seeing (11), we have $I_k=\xi_k+J_k$. Taking it into (18), one gets

$$Z_4 = \varepsilon^2 F'(0)\left[\left(\sum_{k\in\mathbf{Z}}I_k\right)^2 - \frac{1}{2}\sum_{k\in\mathbf{Z}}\xi_k^2 - \sum_{k\in\mathbf{Z}}\xi_k J_k - \frac{1}{2}\sum_{k\in\mathbf{Z}}J_k^2\right] \tag{19}$$

Choosing $\sum_{k\in\mathbf{Z}}\xi_k J_k$ from Z_4, we have

$$\{\sum_{k\in\mathbf{Z}}\xi_k J_k,\psi_{k_1}\psi_{k_2}\psi_{k_3}\bar{\psi}_{k_4}\bar{\psi}_{k_5}\bar{\psi}_{k_6}\} = \mathrm{i}(\xi_{k_1}+\xi_{k_2}+\xi_{k_3}-\xi_{k_4}-\xi_{k_5}-\xi_{k_6})\psi_{k_1}\psi_{k_2}\psi_{k_3}\bar{\psi}_{k_4}\bar{\psi}_{k_5}\bar{\psi}_{k_6} \tag{20}$$

We expect that (20) is exploited to eliminate some terms of R_6, so the problem of the small divisor $\xi_{k_1}+\xi_{k_2}+\xi_{k_3}-\xi_{k_4}-\xi_{k_5}-\xi_{k_6}$ should be dealt in advance. Therefore, it's necessary to consider the following Diophantine condition: there exist $\alpha,\gamma>0$ such that for any N large enough and $\boldsymbol{k}=(k_1,k_2,k_3,k_4,k_5,k_6)\in\mathbf{Z}^6$, one has

$$|\xi_{k_1}+\xi_{k_2}+\xi_{k_3}-\xi_{k_4}-\xi_{k_5}-\xi_{k_6}|\geqslant\frac{\gamma}{N^\alpha} \tag{21}$$

where $\boldsymbol{k}$ contains at most two large indexes (*i. e.*, larger than $N+1$).

We decompose $R_6=R_6^N+\widetilde{R}_6$ as follows

$$R_6^N=-\mathrm{i}\varepsilon^4\sum_{\substack{\boldsymbol{k}\in M_6\\ \mu_3(\boldsymbol{k})\geqslant N}}\widetilde{Z}_{6,\boldsymbol{k}}\Big(\sum_{j=1}^{3}\delta_{k_j,\mathrm{k}}-\sum_{j=4}^{6}\delta_{k_j,\mathrm{k}}\Big)\psi_{k_1}\psi_{k_2}\psi_{k_3}\bar{\psi}_{k_4}\bar{\psi}_{k_5}\bar{\psi}_{k_6} \tag{22}$$

$$\widetilde{R}_6=-\mathrm{i}\varepsilon^4\sum_{\substack{\boldsymbol{k}\in M_6\\ \mu_3(\boldsymbol{k})<N}}\widetilde{Z}_{6,\boldsymbol{k}}\Big(\sum_{j=1}^{3}\delta_{k_j,\mathrm{k}}-\sum_{j=4}^{6}\delta_{k_j,\mathrm{k}}\Big)\psi_{k_1}\psi_{k_2}\psi_{k_3}\bar{\psi}_{k_4}\bar{\psi}_{k_5}\bar{\psi}_{k_6} \tag{23}$$

where $\mu_3(\boldsymbol{k})$ denotes the third largest number among $\{|k_1|,|k_2|,|k_3|,|k_4|,|k_5|,|k_6|\}$, R_6^N contains terms with at least three large indexes and $\widetilde{R}_6$ includes terms with at most two large indexes. And then, we define

$$\widetilde{\Phi}_{\mathrm{k},6}=\sum_{\substack{\boldsymbol{k}\in M_6\\ \mu_3(\boldsymbol{k})<N}}A_{6,\boldsymbol{k}}\psi_{k_1}\psi_{k_2}\psi_{k_3}\bar{\psi}_{k_4}\bar{\psi}_{k_5}\bar{\psi}_{k_6}.$$

Computing

$$\widetilde{R}_6+\Big\{-\varepsilon^2F'(0)\sum_{k\in\mathbf{Z}}\xi_kJ_k,\widetilde{\Phi}_{\mathrm{k},6}\Big\}=0 \tag{24}$$

$$\widetilde{\Phi}_{\mathrm{k},6}=\sum_{\substack{\boldsymbol{k}\in M_6\\ \mu_3(\boldsymbol{k})<N}}\frac{\varepsilon^2\widetilde{Z}_{6,\boldsymbol{k}}\big(\sum_{j=1}^{3}\delta_{k_j,\mathrm{k}}-\sum_{j=4}^{6}\delta_{k_j,\mathrm{k}}\big)}{F'(0)(\xi_{k_4}+\xi_{k_5}+\xi_{k_6}-\xi_{k_1}-\xi_{k_2}-\xi_{k_3})}\psi_{k_1}\psi_{k_2}\psi_{k_3}\bar{\psi}_{k_4}\bar{\psi}_{k_5}\bar{\psi}_{k_6} \tag{25}$$

follows. In view of (12)~(15) and (25), we define the approximate integral of motion as

$$\Phi_{\mathrm{k}}^{(6)}=\Phi_{\mathrm{k},2}+\Phi_{\mathrm{k},4}+\Phi_{\mathrm{k},6}+\widetilde{\Phi}_{\mathrm{k},6}+L_{\chi_4}\widetilde{\Phi}_{\mathrm{k},6} \tag{26}$$

The action result of motion integral is as follows.

Lemma 2.1 Given $\mathrm{k}\in\mathbf{Z}$, considering (7) (8) (26) and the Diophantine condition (21), we have

$$\{H,\Phi_{\mathrm{k}}^{(6)}\}=R_6^N+R_{\geqslant 8} \tag{27}$$

where R_6^N is defined as (22), $R_{\geqslant 8}$ denotes terms of degree at least 8 and

$$R_{\geqslant 8}=\sum_{j=4}^{\infty}\{H_{2j},\Phi_{\mathrm{k},2}\}+\sum_{j=3}^{\infty}\{H_{2j},\Phi_{\mathrm{k},4}\}+\sum_{j=2}^{\infty}\{H_{2j},\Phi_{\mathrm{k},6}\}+\Big\{-\frac{\varepsilon^2F'(0)}{2}\sum_{k\in\mathbf{Z}}J_k^2,\widetilde{\Phi}_{\mathrm{k},6}\Big\}+\sum_{j=3}^{\infty}\{H_{2j},\widetilde{\Phi}_{\mathrm{k},6}\}+\sum_{j=2}^{\infty}\{H_{2j},L_{\chi_4}\widetilde{\Phi}_{\mathrm{k},6}\} \tag{28}$$

Proof We compute

$$\Big\{\Big(\sum_{k\in\mathbf{Z}}I_k\Big)^2,\widetilde{\Phi}_{\mathrm{k},6}\Big\}=0,\ \Big\{\sum_{k\in\mathbf{Z}}\xi_k^2,\widetilde{\Phi}_{\mathrm{k},6}\Big\}=0 \tag{29}$$

noting that the initial value ξ_k is a constant. Considering (19) and (24), one gets

$$R_6+\{Z_4,\widetilde{\Phi}_{\mathrm{k},6}\}=\Big\{-\frac{\varepsilon^2F'(0)}{2}\sum_{k\in\mathbf{Z}}J_k^2,\widetilde{\Phi}_{\mathrm{k},6}\Big\}+R_6^N \tag{30}$$

combining (29) and (30), Lemma 2.1 can be easily proved by using Lemma 4.1 in Ref. [12].

Then we estimate the remainder terms R_6^N and $R_{\geqslant 8}$.

Lemma 2.2 Given $\mathrm{k}\in\mathbf{Z}$, R_6^N in (22), $R_{\geqslant 8}$ in (28) and $\frac{1}{2}<s<1$, there exists a constant $C>0$ *s. t.* for $N>0$ and $\alpha,\gamma>0$,

$$|R_6^N|\leqslant C\frac{\varepsilon^4\langle\mathrm{k}\rangle^{-2s}}{N^s}\|z\|_s^4 \tag{31}$$

$$|R_{\geqslant 8}|\leqslant C\varepsilon^6N^\alpha\langle\mathrm{k}\rangle^{-2s}\|z\|_s^6 \tag{32}$$

Proof For any $\mathrm{k}\in\mathbf{Z}$, define

$$\Delta_1=\Big\{(k_2,k_3,k_4,k_5,k_6)\in\mathbf{Z}^5\,\Big|\,\begin{smallmatrix}k_2+k_3-k_4-k_5-k_6=-\mathrm{k}\\ k_2^2+k_3^2-k_4^2-k_5^2-k_6^2=-\mathrm{k}^2\end{smallmatrix},\ \mu_3(\mathrm{k},k_2,k_3,k_4,k_5,k_6)\geqslant N\Big\},$$

$$\Delta_2=\Big\{(k_1,k_2,k_3,k_4,k_5)\in\mathbf{Z}^5\,\Big|\,\begin{smallmatrix}k_1+k_2+k_3-k_4-k_5=\mathrm{k}\\ k_1^2+k_2^2+k_3^2-k_4^2-k_5^2=\mathrm{k}^2\end{smallmatrix},\ \mu_3(k_1,k_2,k_3,k_4,k_5,\mathrm{k})\geqslant N\Big\}.$$

In terms of (12) (16) and (22), for $j \in \{1,\cdots,6\}$, there exists at least an index k_j equal to k in each monomial of R_6^N. We have

$$|R_6^N| \preceq \underbrace{\left|\varepsilon^4 \sum_{\Delta_1} \psi_{\mathrm{k}} \psi_{k_2} \psi_{k_3} \bar{\psi}_{k_4} \bar{\psi}_{k_5} \bar{\psi}_{k_6}\right|}_{\mathrm{PI}} + \underbrace{\left|\varepsilon^4 \sum_{\Delta_2} \psi_{k_1} \psi_{k_2} \psi_{k_3} \bar{\psi}_{k_4} \bar{\psi}_{k_5} \bar{\psi}_{\mathrm{k}}\right|}_{\mathrm{PII}} \tag{33}$$

where "$\preceq$" denotes that there exists a positive constant C independent of N for $a \preceq b$ (but maybe depends on γ) such that $a \leqslant Cb$.

For PI, one has

$$\mathrm{PI}^2 = \left|\varepsilon^4 \sum_{\Delta_1} \psi_{\mathrm{k}} \psi_{k_2} \psi_{k_3} \bar{\psi}_{k_4} \bar{\psi}_{k_5} \bar{\psi}_{k_6}\right|^2 \tag{34}$$

Due to the null momentum condition, there exists at least an index $l \in \{2,3,4,5,6\}$ s. t. $|k_l| \geqslant \frac{|\mathrm{k}|}{5}$. Then both $|\psi_{\mathrm{k}}|^2$ and $|\psi_{k_l}|^2$ are of order $\langle \mathrm{k} \rangle^{-2s}$ thanks to scaling. In view of the definition of Δ_1, $\{k_2, k_3, k_4, k_5, k_6\} \backslash k_l$ contains at least one large index. Without losing generality, we assume that $k_l = k_2$ and k_4 is the large index in PI. Then, (34) can be rewritten as

$$\mathrm{PI}^2 = \left|\varepsilon^4 \sum_{\Delta_1} \psi_{\mathrm{k}} \psi_{k_2} \langle k_3 \rangle^s \psi_{k_3} \bar{\psi}_{k_4} \langle k_5 \rangle^s \bar{\psi}_{k_5} \langle k_6 \rangle^s \bar{\psi}_{k_6} \frac{1}{\langle k_3 \rangle^s \langle k_5 \rangle^s \langle k_6 \rangle^s}\right|^2 \tag{35}$$

Exploiting Cauchy inequality for (35), we have

$$\mathrm{PI}^2 \preceq \varepsilon^8 \langle \mathrm{k} \rangle^{-4s} \sum_{\Delta_1} \langle k_3 \rangle^{2s} |\psi_{k_3}|^2 |\bar{\psi}_{k_4}|^2 \langle k_5 \rangle^{2s} \cdot |\bar{\psi}_{k_5}|^2 \langle k_6 \rangle^{2s} |\bar{\psi}_{k_6}|^2 \sum_{\Delta_1} \frac{1}{\langle k_3 \rangle^{2s} \langle k_5 \rangle^{2s} \langle k_6 \rangle^{2s}} \tag{36}$$

Due to

$$\begin{cases} k_2 + k_3 - k_4 - k_5 - k_6 = -\mathrm{k} \\ k_2^2 + k_3^2 - k_4^2 - k_5^2 - k_6^2 = -\mathrm{k}^2, \end{cases}$$

if k_3, k_5, k_6 are fixed, then one gets (k_2, k_4) has at most four conditions. Therefore, we have

$$\sum_{\Delta_1} \frac{1}{\langle k_3 \rangle^{2s} \langle k_5 \rangle^{2s} \langle k_6 \rangle^{2s}} \preceq 4 \sum_{k_3, k_5, k_6} \frac{1}{\langle k_3 \rangle^{2s} \langle k_5 \rangle^{2s} \langle k_6 \rangle^{2s}} \tag{37}$$

Thanks to $\frac{1}{2} < s < 1$, we obtain (37) is bounded by a constant. Then, as a result of assuming k_4 is the large index, (36) becomes

$$\mathrm{PI}^2 \preceq \varepsilon^8 \langle \mathrm{k} \rangle^{-4s} \sum_{\Delta_1} \langle k_3 \rangle^{2s} |\psi_{k_3}|^2 \frac{\langle k_4 \rangle^{2s}}{N^{2s}} |\bar{\psi}_{k_4}|^2 \cdot \langle k_5 \rangle^{2s} |\bar{\psi}_{k_5}|^2 \langle k_6 \rangle^{2s} |\bar{\psi}_{k_6}|^2 \preceq \frac{\varepsilon^8 \langle \mathrm{k} \rangle^{-4s}}{N^{2s}} \|z\|_s^8 \tag{38}$$

So, one has

$$\mathrm{PI} \preceq \frac{\varepsilon^4 \langle \mathrm{k} \rangle^{-2s}}{N^s} \|z\|_s^4 \tag{39}$$

Similarly, we can apply the proof of PI to the estimate of PII and obtain the same results as (39). Then (31) is proved.

Recall the construction of the approximate integral of motion *i. e.*, (12)～(15) and (25). For integer $n \geqslant 1$ and finite order monomial $\psi_{k_1} \cdots \psi_{k_n} \bar{\psi}_{k_{n+1}} \cdots \bar{\psi}_{k_{2n}}$ in $\Phi_{\mathrm{k}}^{(6)}$, R_6^N and $R_{\geqslant 8}$, there exists $l_1, l_2 \in \{1,\cdots,2n\}$ and $C_1, C_2 > 0$ s. t. $|k_{l_1}| \geqslant \frac{|\mathrm{k}|}{C_1}$, $|k_{l_2}| \geqslant \frac{|\mathrm{k}|}{C_2}$.

For $R_{\geqslant 8}$ in (28), we define

$$R_{\geqslant 8,1} = \sum_{j=4}^{\infty} \{H_{2j}, \Phi_{\mathrm{k},2}\} + \sum_{j=3}^{\infty} \{H_{2j}, \Phi_{\mathrm{k},4}\} + \sum_{j=2}^{\infty} \{H_{2j}, \Phi_{\mathrm{k},6}\}.$$

Referring to the proof process of (31), we have

$$|R_{\geqslant 8,1}| \preceq \varepsilon^6 \langle \mathrm{k} \rangle^{-2s} \|z\|_s^6 \tag{40}$$

Due to $I_k = \xi_k + J_k$ in (11), we take $J_k = O(\xi_k^2)$. J_k is of order ε^2 as a result of the scale transformation. By computing

$$\left\{-\frac{\varepsilon^2 F'(0)}{2} \sum_{k \in \mathbf{Z}} J_k^2, \widetilde{\Phi}_{\mathrm{k},6}\right\} = \mathrm{i}\varepsilon^4 \sum_{\boldsymbol{k} \in M_6} \frac{\widetilde{Z}_{6,\boldsymbol{k}} \left(\sum_{j=1}^3 \delta_{k_j,\mathrm{k}} - \sum_{j=4}^6 \delta_{k_j,\mathrm{k}}\right)}{\xi_{k_1} + \xi_{k_2} + \xi_{k_3} - \xi_{k_4} - \xi_{k_5} - \xi_{k_6}} \cdot \sum_{q=1}^{6} \left(\sum_{m=1}^{3} \delta_{k_m,k_q} - \sum_{m=4}^{6} \delta_{k_m,k_q}\right) J_{k_q} \psi_{k_1} \psi_{k_2} \psi_{k_3} \bar{\psi}_{k_4} \bar{\psi}_{k_5} \bar{\psi}_{k_6},$$

we obtain

$$\left|\left\{-\frac{\varepsilon^2 F'(0)}{2} \sum_{k \in \mathbf{Z}} J_k^2, \widetilde{\Phi}_{\mathrm{k},6}\right\}\right| \preceq \varepsilon^6 N^{\alpha} \langle \mathrm{k} \rangle^{-2s} \|z\|_s^6 \tag{41}$$

Let

$$R_{\geqslant 8,2} = \sum_{j=3}^{\infty} \{H_{2j}, \widetilde{\Phi}_{\mathrm{k},6}\} + \sum_{j=2}^{\infty} \{H_{2j}, L_{\chi_4} \widetilde{\Phi}_{\mathrm{k},6}\}.$$

we have

$$\left|\sum_{j=3}^{\infty} \{H_{2j}, \widetilde{\Phi}_{\mathrm{k},6}\}\right| \preceq \varepsilon^6 N^{\alpha} \langle \mathrm{k} \rangle^{-2s} \|z\|_s^8,$$

$$\left|\sum_{j=2}^{\infty}\{H_{2j},L_{\chi_4}\ \widetilde{\Phi}_{\mathrm{k},6}\}\right|\leq\varepsilon^6 N^{\alpha}\ \langle\mathrm{k}\rangle^{-2s}\ \|z\|_s^8.$$

Then we obtain

$$|R_{\geqslant 8,2}|\leq\varepsilon^6 N^{\alpha}\ \langle\mathrm{k}\rangle^{-2s}\|z\|_s^8 \tag{42}$$

Because of scaling, $\|z\|_s$ is of order 1, by recalling (40)～(42) one has $|R_{\geqslant 8}|\leq\varepsilon^6 N^{\alpha}\ \langle\mathrm{k}\rangle^{-2s}\ \|z\|_s^6$. Then we get (32) and end the proof.

Concurrently, we could obtain the following result.

Lemma 2.3 Given $\mathrm{k}\in\mathbf{Z}$ and $\frac{1}{2}<s<1$, there exists a constant $C>0$ *s. t.* for $\alpha,\gamma>0$ and $N>0$, $\varepsilon^2 N^{\alpha}<1$,

$$|\Phi_{\mathrm{k}}^{(6)}-|\psi_{\mathrm{k}}|^2|\leqslant C\varepsilon^2 N^{\alpha}\ \langle\mathrm{k}\rangle^{-2s}\|z\|_s^2 \tag{43}$$

Proof By recalling the motion integral (26), one gets

$$|\Phi_{\mathrm{k}}^{(6)}-|\psi_{\mathrm{k}}|^2|=|\Phi_{\mathrm{k},4}+\Phi_{\mathrm{k},6}+\widetilde{\Phi}_{\mathrm{k},6}+L_{\chi_4}\widetilde{\Phi}_{\mathrm{k},6}|.$$

According to (12)～(15) and (25), consulting the proof of Lemma 2.2, and noting the order 1 of $\|z\|_s$ thanks to scaling, we have

$$|\Phi_{\mathrm{k}}^{(6)}-|\psi_{\mathrm{k}}|^2|\leq (\varepsilon^2+\varepsilon^4+\varepsilon^2 N^{\alpha}+\varepsilon^4 N^{\alpha})\langle\mathrm{k}\rangle^{-2s}\|z\|_s^2.$$

Due to $\varepsilon^2 N^{\alpha}<1$, we obtain $\varepsilon^4<\varepsilon^4 N^{\alpha}<\varepsilon^2<\varepsilon^2 N^{\alpha}$. Hence

$$|\Phi_{\mathrm{k}}^{(6)}-|\psi_{\mathrm{k}}|^2|\leq\varepsilon^2 N^{\alpha}\ \langle\mathrm{k}\rangle^{-2s}\|z\|_s^2.$$

The proof is end.

3 Measure estimates

In this section, we complete measure estimates with respect to parameters. At the beginning, for $\frac{1}{2}<s<1$ we consider

$$\widetilde{H}(\psi,\bar{\psi})=\sum_{k\in\mathbf{Z}}(1+k^4)|\psi_k|^2,$$

which defines the Gaussian measure

$$\mathrm{d}\mu_g=\frac{\mathrm{e}^{-\widetilde{H}(\psi,\bar{\psi})}\mathrm{d}\psi\mathrm{d}\bar{\psi}}{\int_{H^s}\mathrm{e}^{-\widetilde{H}(\psi,\bar{\psi})}\mathrm{d}\psi\mathrm{d}\bar{\psi}}=\prod_{k\in\mathbf{Z}}\frac{\mathrm{e}^{-(1+k^4)|\psi_k|^2}\mathrm{d}\psi_k\mathrm{d}\bar{\psi}_k}{\int_{\mathbf{C}}\mathrm{e}^{-(1+k^4)|\psi_k|^2}\mathrm{d}\psi_k\mathrm{d}\bar{\psi}_k}.$$

The relevant theory of Gaussian measure could refer to Refs. [12,15,16]. We only consider the measure estimate in the unit sphere as a result of scaling and the initial condition (2). Let S denote the unit sphere *i. e.*, the corresponding region of $\|z\|_s\leqslant 1$. On the basis of definition of μ_g, one has $\mu_g(S)>0$ under $\frac{1}{2}<s<1$. Hence we could redefine the Gaussian measure restricted S as follows. For any set $E\subset S$, define

$$\mu_g|_S(E)=\frac{\mu_g(E)}{\mu_g(S)} \tag{44}$$

Note that $\mu_g|_S(S)=1$. For $\alpha,\gamma>0$ and $N>0$, denote

$$\Omega=\{\psi\in S|\boldsymbol{k}\in M_6,\mu_3(\boldsymbol{k})<N,|\xi_{k_1}+\xi_{k_2}+\xi_{k_3}-\xi_{k_4}-\xi_{k_5}-\xi_{k_6}|<\frac{\gamma}{N^{\alpha}}\}.$$

We have the following lemma.

Lemma 3.1 For any positive γ small enough, $\frac{1}{2}<s<1$ and $\alpha\geqslant 8$, there exists a constant $\widetilde{C}>0$ such that for $N>0$ the measure of Ω is smaller than $\widetilde{C}\gamma$.

Proof For any $\boldsymbol{k}=(k_1,k_2,k_3,k_4,k_5,k_6)\in M_6$, we have

$$k_1+k_2+k_3=k_4+k_5+k_6 \tag{45}$$

$$k_1^2+k_2^2+k_3^2=k_4^2+k_5^2+k_6^2 \tag{46}$$

Suppose that k_1,k_2,k_3,k_4 are bounded by N, k_5 and k_6 have at most four conditions in terms of (45) and (46). Then the number of possible combinations is at most $(2N+1)^4\times 4$ if $\mu_3(\boldsymbol{k})<N$, *i. e.*, $\boldsymbol{k}$ contains at most two large indexes.

Recalling the definition of R_6^N, the terms with $\{k_1,k_2,k_3\}=\{k_4,k_5,k_6\}$ are equal to 0, thus R_6^N contains only terms with $\{k_1,k_2,k_3\}\cap\{k_4,k_5,k_6\}=\varnothing$ by (45) and (46). For $j,i_1,i_2\in\{1,\cdots,6\}$, let $k'=\{k_j\in\boldsymbol{k}\mid\forall k_{i_1},k_{i_2}\in\boldsymbol{k}',k_{i_1}\neq k_{i_2}\}$, and M denote the cardinality of $\boldsymbol{k}'$. We can obtain

$$\mu_g|_S(\Omega)=\frac{\mu_g(\Omega)}{\mu_g(S)}\leqslant\frac{4(2N+1)^4}{\mu_g(S)}\frac{\int_{\Omega}\mathrm{e}^{-\sum_{k_j\in\boldsymbol{k}'}(1+k_j^4)|\psi_{k_j}|^2}\prod_{k_j\in\boldsymbol{k}'}\mathrm{d}\psi\mathrm{d}\bar{\psi}}{\int_{C^M}\mathrm{e}^{-\sum_{k_j\in\boldsymbol{k}'}(1+k_j^4)|\psi_{k_j}|^2}\prod_{k_j\in\boldsymbol{k}'}\mathrm{d}\psi_{k_j}\mathrm{d}\bar{\psi}_{k_j}} \tag{47}$$

Let $\omega=|\xi_{k_1}+\xi_{k_2}+\xi_{k_3}-\xi_{k_4}-\xi_{k_5}-\xi_{k_6}|$ and define a truncation function

$$\tau(x)=\begin{cases}0, & x\geqslant\frac{\gamma}{N^{\alpha}},\\ 1, & x<\frac{\gamma}{N^{\alpha}}.\end{cases}$$

With the transformation $\psi_{k_j}=\rho_{k_j}\mathrm{e}^{\mathrm{i}\theta_{k_j}}$, $\rho_{k_j}=|\psi_{k_j}|$, $\theta_{k_j}\in[0,2\pi]$, (47) is bounded by

$$\frac{4\ (2N+1)^4}{\mu_g(S)}\frac{\int_{\mathbf{R}_+^M}\mathrm{e}^{-\sum_{k_j\in\boldsymbol{k}'}(1+k_j^4)\rho_{k_j}{}^2}\tau(\omega)\prod_{k_j\in\boldsymbol{k}'}\rho_{k_j}\mathrm{d}\rho_{k_j}}{\int_{\mathbf{R}_+^M}\mathrm{e}^{-\sum_{k_j\in\boldsymbol{k}'}(1+k_j^4)\rho_{k_j}{}^2}\prod_{k_j\in\boldsymbol{k}'}\rho_{k_j}\mathrm{d}\rho_{k_j}} \tag{48}$$

where $\omega=|\rho_{k_1}^2+\rho_{k_2}^2+\rho_{k_3}^2-\rho_{k_4}^2-\rho_{k_5}^2-\rho_{k_6}^2|$ according to (9). Taking the transformation $\eta_{k_j}=(1+k_j^4)\rho_{k_j}^2$, (48) becomes

$$\frac{4\ (2N+1)^4}{\mu_g(S)}\frac{\int_{\mathbf{R}_+^M}\mathrm{e}^{-\sum_{k_j\in\boldsymbol{k}'}\eta_{k_j}}\tau(\omega)\prod_{k_j\in\boldsymbol{k}'}\mathrm{d}\eta_{k_j}}{\prod_{k_j\in\boldsymbol{k}'}\int_{\mathbf{R}_+}\mathrm{e}^{-\eta_{k_j}}\mathrm{d}\eta_{k_j}} \tag{49}$$

where

$$\omega=\left|\frac{1}{1+k_1^4}\eta_{k_1}+\frac{1}{1+k_2^4}\eta_{k_2}+\frac{1}{1+k_3^4}\eta_{k_3}-\frac{1}{1+k_4^4}\eta_{k_4}-\frac{1}{1+k_5^4}\eta_{k_5}-\frac{1}{1+k_6^4}\eta_{k_6}\right|.$$

Let $\widetilde{k}$ denote the index corresponding to the smallest element among $\{|k_j|\}_{j=1}^6$. We consider that $(a_j)_{j=1}^6\in\mathbf{Z}^6$ and define $A=\sum_{j=1}^{6}\frac{a_j}{1+k_j^4}\eta_{k_j}$, where $a_i=\{0,1,2,3\}$ for $i\in\{1,2,3\}$, $a_l=\{0,-1,-2,-3\}$ for $l\in\{4,5,6\}$, $\sum_{j=1}^6|a_j|=6$ and $\widetilde{a}\neq0$ denotes the correspondent coefficient of $\widetilde{k}$. Then, let $\widetilde{A}=\sum_{\widetilde{k}\neq k_j}\frac{a_j}{1+k_j^4}\eta_{k_j}$. Clearly,

$$\int_{\mathbf{R}_+}\mathrm{e}^{-\eta_{k_j}}\ \mathrm{d}\eta_{k_j}=1.$$

Afterwards, there exist constants $C,\widetilde{C}>0$ such that (49) is bounded by

$$\frac{4\ (2N+1)^4}{\mu_g(S)}\prod_{k_j\in\boldsymbol{k}'\backslash\widetilde{k}}\int_{\mathbf{R}_+}\mathrm{e}^{-\eta_{k_j}}\ \mathrm{d}\eta_{k_j}\cdot\int_{\frac{1+\widetilde{k}^4}{\widetilde{a}}(-\frac{\gamma}{N^\alpha}-\widetilde{A})}^{\frac{1+\widetilde{k}^4}{\widetilde{a}}(\frac{\gamma}{N^\alpha}-\widetilde{A})}\mathrm{e}^{-\eta_{\widetilde{k}}}\ \mathrm{d}\eta_{\widetilde{k}}\leqslant\frac{4\ (2N+1)^4}{\mu_g(S)}\prod_{k_j\in\boldsymbol{k}'\backslash\widetilde{k}}\int_{\mathbf{R}_+}\mathrm{e}^{-\eta_{k_j}}\ \mathrm{d}\eta_{k_j}\int_{\frac{1+\widetilde{k}^4}{\widetilde{a}}(-\frac{\gamma}{N^\alpha}-\widetilde{A})}^{\frac{1+\widetilde{k}^4}{\widetilde{a}}(\frac{\gamma}{N^\alpha}-\widetilde{A})}\mathrm{d}\eta_{\widetilde{k}}\leqslant\frac{CN^4\ \widetilde{k}^4\gamma}{\mu_g(S)N^\alpha}\leqslant\frac{C\gamma}{\mu_g(S)N^{\alpha-8}}\leqslant\widetilde{C}\gamma.$$

Noticing that $|\widetilde{k}|<N$ and $\alpha\geqslant8$, Lemma 3.1 is proved.

4　The proof of Theorem 1.1

Assume that $\|z\|_s\leqslant8$ thanks to scaling. Applying Lemma 2.2, one has

$$|R_6^N|\leq\frac{\varepsilon^4\ \langle\mathrm{k}\rangle^{-2s}}{N^s},\ |R_{\geqslant8}|\leq\varepsilon^6N^\alpha\ \langle\mathrm{k}\rangle^{-2s}.$$

Computing $\frac{\varepsilon^4}{N^s}=\varepsilon^6N^\alpha$, we take $N=\varepsilon^{-\frac{2}{\alpha+s}}$. Then we obtain $|\{H,\Phi_{\mathrm{k}}^{(6)}\}|\leq\varepsilon^{4+\frac{2s}{\alpha+s}}\langle\mathrm{k}\rangle^{-2s}$. In terms of Lemma 3.1, we choose $\alpha=8$. Particularly, we take $N=\varepsilon^{-\frac{2}{9}}$. One has

$$|\{H,\Phi_{\mathrm{k}}^{(6)}\}|\leq\varepsilon^{4+\frac{2}{9}s}\langle\mathrm{k}\rangle^{-2s} \tag{50}$$

Then, by adopting Lemma 2.3, we have

$$|\Phi_{\mathrm{k}}^{(6)}-|\psi_{\mathrm{k}}|^2|\leq\varepsilon^{\frac{2}{9}}\langle\mathrm{k}\rangle^{-2s} \tag{51}$$

Due to

$$|\dot{\Phi}_{\mathrm{k}}^{(6)}|=|\{H,\Phi_{\mathrm{k}}^{(6)}\}|,\ \Phi_{\mathrm{k}}^{(6)}(\psi(t))-\Phi_{\mathrm{k}}^{(6)}(\psi(0))=\int_0^t\dot{\Phi}_{\mathrm{k}}^{(6)}(\psi(\upsilon))\mathrm{d}\upsilon,$$

one has

$$|\Phi_{\mathrm{k}}^{(6)}(\psi(t))-\Phi_{\mathrm{k}}^{(6)}(\psi(0))|\leqslant\int_0^t|\{H,\Phi_{\mathrm{k}}^{(6)}\}|\mathrm{d}\upsilon\leqslant t|\{H,\Phi_{\mathrm{k}}^{(6)}\}|.$$

In terms of (50), one gets

$$|\Phi_{\mathrm{k}}^{(6)}(\psi(t))-\Phi_{\mathrm{k}}^{(6)}(\psi(0))|\leq t\varepsilon^{4+\frac{2}{9}s}\langle\mathrm{k}\rangle^{-2s}.$$

To prove Theorem 1.1, we have

$$||\psi_{\mathrm{k}}(t)|^2-|\psi_{\mathrm{k}}(0)|^2|\leqslant|\Phi_{\mathrm{k}}^{(6)}(\psi(t))-\Phi_{\mathrm{k}}^{(6)}(\psi(0))|+|\Phi_{\mathrm{k}}^{(6)}(\psi(t))-|\psi_{\mathrm{k}}(t)|^2|+|\Phi_{\mathrm{k}}^{(6)}(\psi(0))-|\psi_{\mathrm{k}}(0)|^2|.$$

Using (50) and (51), we have

$$\langle\mathrm{k}\rangle^{2s}||\psi_{\mathrm{k}}(t)|^2-|\psi_{\mathrm{k}}(0)|^2|\leq t\varepsilon^{4+\frac{2}{9}s}+\varepsilon^{\frac{2}{9}}.$$

There exists a constant $C>0$ *s.t.*

$$\langle\mathrm{k}\rangle^{2s}||\psi_{\mathrm{k}}(t)|^2-|\psi_{\mathrm{k}}(0)|^2|\leqslant1,$$

for $t\leqslant C\varepsilon^{-(4+\frac{2}{9}s)}$. Finally, we restore to the original order ε^2 of $|\psi_{\mathrm{k}}|^2$ and fix $\gamma>0$ small enough in Lemma 3.1. Because k can be taken arbitrarily in $\mathbf{Z}$, Theorem 1.1 is proved.

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